STABILITY OF QUADRATIC FUNCTIONAL EQUATIONS IN ŠERSTNEV PROBABILISTIC NORMED SPACES

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In this paper, we prove the Hyers-Ulam-Rassias stability of the following quadratic functional equations in Š*erstnev probabilistic normed space endowed with* $\Pi_{\mathcal{M}}$ *triangle function:*

$$f(x+y)+f(x-y)=2f(x)+2f(y),$$
$$f(ax+by)+f(ax-by)=2a^2f(x)+2b^2f(y)$$

for nonzero real numbers a, b *with* $a \neq \pm 1$. *More precisely, we show under some suitable conditions that an approximately quadratic function can be approximated by a quadratic mapping in above mentioned spaces.*

Keywords: Šerstnev probabilistic normed space, quadratic functional equation, Hyers-Ulam-Rassias stability.

MSC2010: 39B82, 39B72, 54E70.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [32] concerning the stability of group homomorphisms. Hyers [16] gave a first affirmative answer to the question of Ulam on approximately additive mappings for Banach spaces. Hyers' theorem was generalized by by Aoki [6] for additive mapping and by Rassias [24] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [24] has provided a lot of influence in the development of what we now call *Hyers-Ulam-Rassias of functional equations*. Găvruţă [12] provided a further generalization in the spirit of Rassias' stability theorem. Later there have been proved several new results on stability of various classes of functional equations in the Hyers-Ulam sense (see [2, 11, 17, 19, 21, 25, 26] and the references cited therein); as well as various stability of different functional equations in Menger probabilistic normed spaces and random normed spaces has been recently studied (cf. [7, 10, 14, 15, 30]). In [13], the authors established generalized Ulam-Hyers stability of Jensen functional equation in Šerstnev probabilistic normed spaces (briefly, Šerstnev PN-spaces). In particular, they proved that if an approximate Jensen mapping in a Šerstnev PN-space is continuous at a point then can be approximate it by

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an anywhere continuous Jensen mapping. As a version of Schwaiger [27], they also showed that if every approximate Jensen type mapping from natural numbers into a Šerstnev PN-space can be approximate by an additive mapping then the norm of Šerstnev PN-space is complete.

In this paper, we consider the following functional equations [22]:

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1}$$

$$f(ax+by)+f(ax-by)=2a^2f(x)+2b^2f(y) \tag{2}$$

for nonzero real numbers a, b with $a \neq \pm 1$ and prove Hyers-Ulam-Rassias stability of the functional equation (1) and (2) in Šerstnev probabilistic normed space endowed with $\Pi_{\mathcal{M}}$ triangle function. More precisely, we show under some suitable conditions that an approximately quadratic function can be approximated by a quadratic mapping in Šerstnev probabilistic normed spaces.

The functional equation (1) is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. It plays a fundamental role in the study of inner product spaces [1, 5, 18], and its solutions are related to symmetric biadditive mapping (see [1, 20]). The Hyers-Ulam stability of equation (1) was proved by Skof [31] for mappings from a normed space to a Banach space. Cholewa [8] noticed that Skof's theorem remains true if the domain is replaced by an Abelian group. In 1992, Czerwik [9] gave a generalization of the Skof–Cholewa's result. Later, Lee et. al. [22] proved Hyers-Ulam-Rassias stability of equations (1) and (2) in fuzzy Banach spaces.

The notion of a probabilistic normed space was introduced by Šerstnev [29]. In [3, 4], Alsina et. al. gave a general definition of probabilistic normed space based on the definition of Menger for probabilistic metric spaces [23]. The theory of probabilistic normed spaces is important as a generalization of deterministic result of linear normed spaces and also in the study of random operator equations.

We recall and apply the definition of probabilistic space briefly as given in [29], together with the notation that will be needed [28]. A distance distribution function (briefly, a d.d.f.) is a nondecreasing function F from $\overline{\mathbb{R}}^+$ into $[0,1]$ that satisfies $F(0)=0$ and $F(+\infty)=1$, and is left–continuous on $(0,+\infty)$; here as usual, $\overline{\mathbb{R}}^+ := [0,+\infty]$. The space of distance distribution functions will be denoted by Δ^+, and the set of all F in Δ^+ for which $\lim\limits_{t\to+\infty^-} F(t)=1$ by D^+. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all t in $\overline{\mathbb{R}}^+$. For any $a \geq 0$, ε_a^+ is the distance distribution function given by

$$\varepsilon_a^+(t) = \begin{cases} 0, & \text{if } t \leq a, \\ 1, & \text{if } t > a. \end{cases} \tag{3}$$

The space Δ^+ can be metrized in several ways [28], but we will here adopt the Sibley metric d_s. If F, G are d.f.'s and h is in $]0,1[$, let $(F,G;h)$ denote the condition: $G(x) \leq F(x+h)+h$, for all $x \in]0, \frac{1}{h}[$. Then the Sibley metric d_s is defined by

$$d_s(F,G) := \inf\big\{h \in]0,1[\,\big|\, \text{both } (F,G;h) \text{ and } (G,F;h)\big\}.$$

In particular, under the usual pointwise ordering of functions, ε_0 is the maximal element of Δ^+. A triangle function is a binary operation on Δ^+, namely, a function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ that is associative, commutative, nondecreasing in each place, and has ε_0 as identity. Moreover, a triangle function is continuous if it is continuous in the metric space (Δ^+, d_s).

Typical continuous triangle functions are $\Pi_T(F,G)(x) = \sup_{s+t=x} T(F(s), G(t))$, and $\Pi_{T^*}(F,G)(x) = \inf_{s+t=x} T^*(F(s), G(t))$. Here T is a continuous t-norm, that is, a continuous binary operation on $[0,1]$ that is commutative, associative, nondecreasing in each variable and has 1 as identity; T^* is a continuous t-conorm, namely a continuous binary operation on $[0,1]$ which is related to the continuous t-norm T through $T^*(x,y) = 1 - T(1-x, 1-y)$. For example, $T(x,y) = \min(x,y) = M(x,y)$ and $T^*(x,y) = \max(x,y)$ or $T(x,y) = \pi(x,y) = xy$ and $T^*(x,y) = \pi^*(x,y) = x+y-xy$.

Note that $\Pi_{\mathcal{M}}(F,G)(x) = \min\{F(x), G(x)\}$ for $F, G \in \Delta^+$ and $x \in \mathbb{R}^+$.

Definition 1.1. (cf. [14, 15]) *A Probabilistic Normed space (briefly, PN space) is a quadruple (X, ν, τ, τ^*), where X is a real vector space, τ and τ^* are continuous triangle functions with $\tau \leq \tau^*$ and ν is a mapping (the probabilistic norm) from X into Δ^+ such that for every choice of p and q in X the following hold:*
(N1) $\nu_p = \varepsilon_0$ if and only if $p = \theta$ (θ is the null vector in X),
(N2) $\nu_{-p} = \nu_p$,
(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$,
(N4) $\nu_p \leq \tau^*(\nu_{\lambda p}, \nu_{(1-\lambda)p})$ for every $\lambda \in [0,1]$.

A PN space is called Šerstnev space if it satisfies (N1), (N3) and the following condition:

$$\nu_{\alpha p}(t) = \nu_p(\frac{t}{|\alpha|}) \tag{4}$$

holds for every $\alpha \neq 0 \in \mathbb{R}$ and $t > 0$. When there is a continuous t-norm T such that $\tau = \Pi_T$ and $\tau^* = \Pi_{T^*}$, the PN space (X, ν, τ, τ^*) is called Menger PN space (briefly, MPN space), and is denoted by (X, ν, τ).

Let (X, ν, τ) be a MPN space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that

$$\lim_{n\to\infty} \nu(x_n - x)(t) = 1 \tag{5}$$

for all $t > 0$. In this case x is called the limit of $\{x_n\}$.

The sequence $\{x_n\}$ in MPN space (X, ν, τ) is called Cauchy if for each $\varepsilon > 0$ and $\delta > 0$, there exists some n_0 such that $\nu(x_n - x_m)(\delta) > 1 - \varepsilon$ for all $m, n \geq n_0$.

Clearly, every convergent sequence in MPN space is Cauchy. If each Cauchy sequence is convergent in MPN space (X, ν, τ), then (X, ν, τ) is called Menger Probabilistic Banach space (briefly, MPB space).

2. **Stability of quadratic functional equations (1)**

In this section, we prove uniform and nonuniform version of the Hyers-Ulam-Rassias stability of equation (1) in Šerstnev MPN space.

Theorem 2.1. *Let X be a linear space and $(\Upsilon, \nu, \Pi_{\mathcal{M}})$ be a* Šerstnev *MPB space. Let $\varphi : X^2 \to [0, \infty)$ be a control function such that*

$$\tilde{\varphi}_n(x, y) = \{4^{-n-1}\varphi(2^n x, 2^n y)\} \tag{6}$$

converges to zero for all $x, y \in X$. Let $f : X \to \Upsilon$ be a uniformly approximately quadratic function with respect to φ and $f(0) = 0$ in the sense that

$$\lim_{t\to\infty} \nu(f(x + y) + f(x - y) - 2f(x) - 2f(y))(t\varphi(x, y)) = 1 \tag{7}$$

uniformly on $X \times X$. Then $Q(x) := \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to \Upsilon$ such that if for some $\delta > 0, \alpha > 0$

$$\nu(f(x + y) + f(x - y) - 2f(x) - 2f(y))(\delta\varphi(x, y)) \geq \alpha \tag{8}$$

for all $x, y \in X$, then

$$\nu(Q(x) - f(x))(\delta\tilde{\varphi}_n(x, x)) \geq \alpha \tag{9}$$

for all $x, y \in X$. Furthermore, the quadratic mapping $Q : X \to \Upsilon$ is the unique mapping such that

$$\lim_{t\to\infty} \nu(Q(x) - f(x))(t\tilde{\varphi}_n(x, x)) = 1 \tag{10}$$

uniformly on X.

Proof. For a given $\varepsilon > 0$, by (7), we can find some $t_0 \geq 0$ such that

$$\nu(f(x + y) + f(x - y) - 2f(x) - 2f(y))(t\varphi(x, y)) \geq 1 - \varepsilon \tag{11}$$

for all $x, y \in X$ and all $t \geq t_0$. Putting $y = x$ in (11), we obtain

$$\nu(4f(x) - f(2x))(t\varphi(x, x)) \geq 1 - \varepsilon \tag{12}$$

and replacing x by $2^n x$, we get

$$\nu(4^{-n-1}f(2^{n+1}x) - 4^{-n}f(2^n x))(t4^{-n-1}\varphi(2^n x, 2^n x)) \geq 1 - \varepsilon. \tag{13}$$

By passing to a nonincreasing subsequence, if necessary, we may assume that $\{4^{-n-1}\varphi(2^n x, 2^n x)\}$ is nonincreasing.

Thus for each $n > m$, we have

$$\begin{aligned}
&\nu(4^{-m}f(2^m x) - 4^{-n}f(2^n x))(t4^{-m-1}\varphi(2^m x, 2^m x))\\
&= \nu(\sum_{k=m}^{n-1}(4^{-k}f(2^k x) - 4^{-k-1}f(2^{k+1}x)))(t4^{-m-1}\varphi(2^m x, 2^m x))\\
&\geq \Pi_{\mathcal{M}}\{\nu(4^{-m}f(2^m x) - 4^{-m-1}f(2^{m+1}x)),\\
&\nu(\sum_{k=m+1}^{n-1}(4^{-k}f(2^k x) - 4^{-k-1}f(2^{k+1}x)))\}(t4^{-m-1}\varphi(2^m x, 2^m x))\\
&\geq \Pi_{\mathcal{M}}\{1 - \varepsilon, \Pi_{\mathcal{M}}\{\nu(4^{-m-1}f(2^{m+1}x) - 4^{-m-2}f(2^{m+2}x)),\\
&\nu(\sum_{k=m+2}^{n-1}(4^{-k}f(2^k x) - 4^{-k-1}f(2^{k+1}x)))\}(t4^{-m-2}\varphi(2^m x, 2^m x))\}\\
&\geq 1 - \varepsilon.
\end{aligned} \tag{14}$$

It follows from (6) that for a given $\delta > 0$ there is an $n_0 \in \mathbb{N}$ such that

$$t_0 4^{-n-1}\varphi(2^n x, 2^n x) < \delta, \ \forall n \geq n_0. \tag{15}$$

Thus by (14) we deduce that

$$\begin{aligned} &\nu(4^{-m}f(2^m x) - 4^{-n}f(2^n x))(\delta) \\ &\geq \nu(4^{-m}f(2^m x) - 4^{-n}f(2^n x))(t_0 4^{-m-1}\varphi(2^m x, 2^m x))\} \geq 1-\varepsilon. \end{aligned} \tag{16}$$

for each $n \geq n_0$. Thus $\{\frac{f(2^n x)}{4^n}\}$ is Cauchy sequence in Υ. Since $(\Upsilon, \nu, \Pi_{\mathcal{M}})$ is complete, the sequence $\{\frac{f(2^n x)}{4^n}\}$ converges to some point $Q(x) \in \Upsilon$. So, we can define a mapping $Q : X \to \Upsilon$ by $Q(x) := \lim\limits_{n\to\infty} \frac{f(2^n x)}{4^n}$, namely, for each $t > 0$, and $x \in X$,

$$\lim_{n\to\infty} \nu(Q(x) - \frac{f(2^n x)}{4^n})(t) = 1. \tag{17}$$

Let $x, y \in X$. Fix $t > 0$ and $0 < \varepsilon < 1$. Since $\{4^{-n-1}\varphi(2^n x, 2^n x)\}$ converges to zero, there is some $n_1 > n_0$ such that $t_0\varphi(2^n x, 2^n x) < t4^{n+1}$ for all $n \geq n_1$. Hence for each $n \geq n_1$, we have

$$\begin{aligned} &\nu(Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y))(t) \\ &\geq \Pi_{\mathcal{M}}\{\Pi_{\mathcal{M}}\{\nu(Q(x+y) - \frac{f(2^{n+1}(x+y))}{4^{n+1}})(t), \nu(Q(x-y) - \frac{f(2^{n+1}(x-y))}{4^{n+1}})(t)\}, \\ &\Pi_{\mathcal{M}}\{\nu(Q(x) - 4^{-n-1}\cdot 2f(2^{n+1}x))(t), \nu(Q(y) - 4^{-n-1}\cdot 2f(2^{n+1}y))(t), \\ &\nu(f(2^{n+1}(x+y)) + f(2^{n+1}(x-y)) - 2f(2^{n+1}x) - 2f(2^{n+1}y))(4^{n+1}t)\}\}. \end{aligned} \tag{18}$$

The first four terms on the right–hand side of the above inequality tend to 1 as $n \to \infty$, and the fifth term is greater than

$$\nu(f(2^{n+1}(x+y)) + f(2^{n+1}(x-y)) - 2f(2^{n+1}x) - 2f(2^{n+1}y)(t_0\varphi(2^n x, 2^n y))$$

which is greater than or equal to $1 - \varepsilon$. Thus

$$\nu(Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y))(t) \geq 1 - \varepsilon$$

for all $t > 0$. It follows that $\nu(Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y))(t) = 1$ for all $t > 0$. By (N1), we have $Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) = 0$ for all $x \in X$. Hence the mapping $Q : X \to \Upsilon$ is quadratic.

Next, let (8) holds for some positive δ and α. And we can put $m = 0$ and $\alpha = 1 - \varepsilon$ in (16) for all $x \in X$, we get $\nu(f(2^n x) - 4^n f(x))(\delta) \geq \alpha$ for all positive integers $n \geq n_0$. Thus for large enough n, we have

$$\begin{aligned} &\nu(f(x) - Q(x))(\delta 4^{-n-1}\varphi(2^n x, 2^n x)) \geq \\ &\Pi_{\mathcal{M}}\{\nu(f(x) - 4^{-n}f(2^n x)), \nu(4^{-n}f(2^n x) - Q(x))\}(\delta 4^{-n-1}\varphi(2^n x, 2^n x)) \geq \alpha, \end{aligned}$$

therefore

$$\nu(Q(x) - f(x))(\delta\tilde{\varphi}_n(x, x)) \geq \alpha.$$

The existence of uniform limit (10) immediately follows from the proof of the first part of Theorem 2.1. It remains to prove the uniqueness assertion. Let Q' be another quadratic mapping satisfying (1) and (10). Fix $c > 0$. Given $\varepsilon > 0$, by (10) for Q and Q', we can choose some t_0 such that

$$\nu(f(x) - Q(x))(t\tilde{\varphi}_n(x,x)) \geq 1 - \varepsilon, \qquad \nu(f(x) - Q'(x))(t\tilde{\varphi}_n(x,x)) \geq 1 - \varepsilon$$

for all $x \in X$ and $t \geq t_0$. Fix some $x \in X$ and find some integer n_0 such that

$$t_0 4^{-n}\varphi(2^n x, 2^n x) < c,$$

for all $n \geq n_0$. Thus we have

$$\begin{aligned}\nu(Q(x) - Q'(x))(c) &\geq \Pi_{\mathcal{M}}\{\nu(4^{-n}f(2^n x) - Q'(x)), \nu(Q(x) - 4^{-n}f(2^n x))\}(c)\\ &= \Pi_{\mathcal{M}}\{\nu(f(2^n x) - Q'(2^n x)), \nu(Q(2^n x) - f(2^n x))\}(4^n c)\\ &\geq \Pi_{\mathcal{M}}\{\nu(f(2^n x) - Q'(2^n x)), \nu(Q(2^n x) - f(2^n x))\}(t_0\varphi(2^n x, 2^n x))\\ &\geq 1 - \varepsilon.\end{aligned}$$

It follows that $\nu(Q(x) - Q'(x))(c) = 1$ for all $c > 0$. Thus $Q(x) = Q'(x)$ for all $x \in X$. □

Corollary 2.1. *Let X be a linear normed space and $(\Upsilon, \nu, \Pi_{\mathcal{M}})$ be a* Šerstnev *MPB space. Let $\theta \geq 0$ and $0 \leq p < 2$. Suppose that $f : X \to \Upsilon$ is a mapping with $f(0) = 0$ such that*

$$\lim_{t\to\infty} \nu(f(x+y) + f(x-y) - 2f(x) - 2f(y))(t\theta(\|x\|^p + \|y\|^p)) = 1 \tag{19}$$

uniformly on $X \times X$. Then $Q(x) := \lim_{n\to\infty} 4^{-n}f(2^n x)$ exists for all $x \in X$ and defines a quadratic mapping $Q : X \to \Upsilon$ such that if for some $\delta > 0, \alpha > 0$

$$\nu(f(x+y) + f(x-y) - 2f(x) - 2f(y))(\delta\theta(\|x\|^p + \|y\|^p)) \geq \alpha \tag{20}$$

for all $x, y \in X$, then

$$\nu(Q(x) - f(x))(\frac{2^{n(p-2)}}{2}\delta\theta\|x\|^p) \geq \alpha \tag{21}$$

for all $x, y \in X$. Furthermore, the quadratic mapping $Q : X \to \Upsilon$ is the unique mapping such that

$$\lim_{t\to\infty} \nu(Q(x) - f(x))(\frac{2^{n(p-2)}}{2}t\theta\|x\|^p) = 1 \tag{22}$$

uniformly on X.

Proof. Define $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ and apply Theorem 2.1 to get the result. □

We are ready to give our nonuniform version of the Hyers-Ulam-Rassias theorem for equation (1) in Šerstnev MPB space.

Theorem 2.2. *Let X be a linear space and $(Z, \omega, \Pi_{\mathcal{M}})$ be a* Šerstnev *MPN space. Let $\psi : X^2 \to Z$ be a function such that for some $0 < \alpha < 4$*

$$\omega(\psi(2x, 2y))(t) \geq \omega(\alpha\psi(x, y))(t) \tag{23}$$

for all $x, y \in X$ and $t > 0$. Let $(\Upsilon, \nu, \Pi_{\mathcal{M}})$ be a Šerstnev *MPB space and let $f : X \to \Upsilon$ be a ψ-approximately quadratic mapping with $f(0) = 0$ in the sense that*

$$\nu(f(x+y) + f(x-y) - 2f(x) - 2f(y))(t) \geq \omega(\psi(x, y))(t) \tag{24}$$

for all $x, y \in X$ and $t > 0$. Then there exists unique quadratic mapping $Q : X \to \Upsilon$ such that

$$\nu(f(x) - Q(x))(t) \geq \omega(\frac{1}{4}\psi(x,x))(t) \tag{25}$$

for all $x \in X$ and $t > 0$.

Proof. Putting $y = x$ in (24), we obtain

$$\nu(f(2x) - 4f(x))(t) \geq \omega(\psi(x,x))(t) \tag{26}$$

for all $x \in X$ and $t > 0$. Using (23) and induction on n, one can verify that

$$\omega(\psi(2^n x, 2^n x))(t) \geq \omega(\alpha^n \psi(x,x))(t) \tag{27}$$

for all $x \in X$ and $t > 0$. It follows from (26) and (27) that

$$\nu(4^{-n} f(2^n x) - 4^{-n+1} f(2^{n-1}x))((\frac{\alpha^n}{4^n})t) \geq \omega((\frac{1}{\alpha})\psi(x,x))(t). \tag{28}$$

Thus for all $n \geq m \geq 0, x \in X$ and $t > 0$, we have

$$\begin{aligned}
&\nu(4^{-n} f(2^n x) - 4^{-m} f(2^m x))((\frac{\alpha^{m+1}}{4^{m+1}})t) \\
&= \nu(\sum_{k=m+1}^{n} 4^{-k} f(2^k x) - 4^{-k+1} f(2^{k-1}x))((\frac{\alpha^{m+1}}{4^{m+1}})t) \geq \omega((\frac{1}{\alpha})\psi(x,x))(t).
\end{aligned} \tag{29}$$

So we get

$$\nu(4^{-n} f(2^n x) - 4^{-m} f(2^m x))(t) \geq \omega((\frac{1}{\alpha})\psi(x,x))((\frac{4^{m+1}}{\alpha^{m+1}})t). \tag{30}$$

Fix $x \in X$. Thanks to the fact that $\lim_{s\to\infty} \omega((\frac{1}{\alpha})\psi(x,x))(s) = 1$, we deduce that $\{\frac{f(2^n x)}{4^n}\}$ is a Cauchy sequence in Υ. Since $(\Upsilon, \nu, \Pi_{\mathcal{M}})$ is complete, this sequence converges to some point $Q(x) \in \Upsilon$. Using (30) with $m = 0$, we obtain

$$\begin{aligned}
\nu(Q(x) - f(x))(t) &\geq \Pi_{\mathcal{M}}\{\nu(Q(x) - 4^{-n} f(2^n x)), \nu(4^{-n} f(2^n x) - f(x))\}(t) \\
&\geq \Pi_{\mathcal{M}}\{\nu(Q(x) - 4^{-n} f(2^n x)), \omega(\frac{1}{4}\psi(x,x))\}(t).
\end{aligned} \tag{31}$$

Hence

$$\begin{aligned}
\nu(Q(x) - f(x))(t) &\geq \Pi_{\mathcal{M}}\{\lim_{n\to\infty} \nu(Q(x) - 4^{-n} f(2^n x)), \omega(\frac{1}{4}\psi(x,x))\}(t) \\
&= \omega(\frac{1}{4}\psi(x,x))(t).
\end{aligned}$$

It follows from (24) that

$$\begin{aligned}
&\nu(\frac{f(2^n(x+y))}{4^n} + \frac{f(2^n(x-y))}{4^n} - 2\frac{f(2^n x)}{4^n} - 2\frac{f(2^n y)}{4^n})(t) \\
&\geq \omega(\psi(x,y))((\frac{4}{\alpha})^n t).
\end{aligned} \tag{32}$$

Hence we have

$$\begin{aligned}&\nu(Q(x+y)+Q(x-y)-2Q(x)-2Q(y))(t)\\&\geq \Pi_{\mathcal{M}}\{\Pi_{\mathcal{M}}\{\nu(Q(x+y)-\frac{f(2^n(x+y))}{4^n})(t),\nu(Q(x-y)-\frac{f(2^n(x-y))}{4^n})(t)\},\\&\Pi_{\mathcal{M}}\{\nu(Q(x)-2\frac{f(2^nx)}{4^n})(t),\nu(Q(y)-2\frac{f(2^ny)}{4^n})(t),\\&\nu(\frac{f(2^n(x+y))}{4^n}+\frac{f(2^n(x-y))}{4^n}-2\frac{f(2^nx)}{4^n}-2\frac{f(2^ny)}{4^n})(t)\}\}.\end{aligned} \tag{33}$$

By (32) and the fact that $\lim_{n\to\infty}\nu(Q(x)-\frac{f(2^nx)}{4^n})(t)=1$ for all $x\in X$ and $t>0$, each term on the right–hand side tends to 1 as $n\to\infty$. Hence

$$\nu(Q(x+y)+Q(x-y)-2Q(x)-2Q(y))(t)=1. \tag{34}$$

By (N1), it follows that $Q(x+y)+Q(x-y)=2Q(x)+2Q(y)$. The uniqueness of Q can be proved in a similar fashion as in the proof of Theorem 2.1. □

3. **Stability of quadratic functional equations (2)**

In this section, we prove uniform and nonuniform version of the Hyers-Ulam-Rassias stability of equation (2) in Šerstnev MPN space. From now on, we suppose that a, b are nonzero real numbers with $a\neq\pm1$.

Lemma 3.1. (cf. [22]). *Let V and W be real vector spaces. If a mapping $f:V\to W$ satisfies*

$$f(ax+by)+f(ax-by)=2a^2f(x)+2b^2f(y)$$

for all $x,y\in V$, then the mapping $f:V\to W$ is quadratic, i.e.,

$$f(x+y)+2f(x-y)=2f(x)+2f(y)$$

holds for all $x,y\in V$.

Theorem 3.1. *Let X be a linear space and $(\Upsilon,\nu,\Pi_{\mathcal{M}})$ be a* Šerstnev *MPB space. Let $\varphi:X^2\to[0,\infty)$ be a control function such that*

$$\tilde{\varphi}_n(x,0)=\{a^{-2n-2}\varphi(a^nx,0)\} \tag{35}$$

converges to zero for all $x,y\in X$. Let $f:X\to\Upsilon$ be a uniformly approximately quadratic function with respect to φ and $f(0)=0$ in the sense that

$$\lim_{t\to\infty}\nu(f(ax+by)+f(ax-by)-2a^2f(x)-2b^2f(y))(t\varphi(x,y))=1 \tag{36}$$

uniformly on $X\times X$. Then $Q(x):=\lim_{n\to\infty}\frac{f(a^nx)}{a^{2n}}$ exists for each $x\in X$ and defines a quadratic mapping $Q:X\to\Upsilon$ such that if for some $\delta>0,\alpha>0$

$$\nu(f(ax+by)+f(ax-by)-2a^2f(x)-2b^2f(y))(\delta\varphi(x,y))\geq\alpha \tag{37}$$

for all $x,y\in X$, then

$$\nu(Q(x)-f(x))(\delta\tilde{\varphi}_n(x,0))\geq\alpha \tag{38}$$

for all $x, y \in X$. *Furthermore, the quadratic mapping* $Q : X \to \Upsilon$ *is the unique mapping such that*

$$\lim_{t\to\infty} \nu(Q(x) - f(x))(t\tilde{\varphi}_n(x, 0)) = 1 \tag{39}$$

uniformly on X.

Proof. For a given $\varepsilon > 0$, by (36), we can choose some $t_0 \geq 0$ such that

$$\nu(f(ax + by) + f(ax - by) - 2a^2 f(x) - 2b^2 f(y))(t\varphi(x, y)) \geq 1 - \varepsilon \tag{40}$$

for all $x, y \in X$ and all $t \geq 2t_0$. Letting $y = 0$ in (40), we get

$$\nu(2f(ax) - 2a^2 f(x))(t\varphi(x, 0)) \geq 1 - \varepsilon \tag{41}$$

and replacing x by $a^n x$, we get

$$\nu(a^{-2n-2} f(a^{n+1}x) - a^{-2n} f(a^n x))(ta^{-2n-2}\varphi(a^n x, 0)) \geq 1 - \varepsilon \tag{42}$$

for all $x, y \in X$ and all $t \geq 2t_0$. By passing to a nonincreasing subsequence, if necessary, we may assume that $\{a^{-2n-2}\varphi(a^n x, 0)\}$ is nonincreasing.

Thus for each $n > m$, we have

$$\begin{aligned}
&\nu(a^{-2m} f(a^m x) - a^{-2n} f(a^n x))(ta^{-2m-2}\varphi(a^m x, 0)) \\
&= \nu(\sum_{k=m}^{n-1} (a^{-2k} f(a^k x) - a^{-2k-2} f(a^{k+1}x)))(ta^{-2m-2}\varphi(a^m x, 0)) \\
&\geq \Pi_{\mathcal{M}}\{\nu(a^{-2m} f(a^m x) - a^{-2m-2} f(a^{m+1}x)), \\
&\nu(\sum_{k=m+1}^{n-1} (a^{-2k} f(a^k x) - a^{-2k-2} f(a^{k+1}x)))\}(ta^{-2m-2}\varphi(a^m x, 0)) \\
&\geq 1 - \varepsilon.
\end{aligned} \tag{43}$$

It follows from (35) that for a given $\delta > 0$ there is an $n_0 \in \mathbb{N}$ such that

$$t_0 a^{-2n-2}\varphi(a^n x, 0) < \delta, \ \forall n \geq n_0. \tag{44}$$

Thus by (43) we deduce that

$$\begin{aligned}
&\nu(a^{-2m} f(a^m x) - a^{-2n} f(a^n x))(\delta) \\
&\geq \nu(a^{-2m} f(a^m x) - a^{-2n} f(a^n x))(t_0 a^{-2m-2}\varphi(a^m x, 0))\} \geq 1 - \varepsilon.
\end{aligned} \tag{45}$$

for each $n \geq n_0$. Thus the sequence $\{\frac{f(a^n x)}{a^{2n}}\}$ is Cauchy in Υ. Since $(\Upsilon, \nu, \Pi_{\mathcal{M}})$ is complete, the sequence $\{\frac{f(a^n x)}{a^{2n}}\}$ converges to some point $Q(x) \in \Upsilon$. So we can define a mapping $Q : X \to \Upsilon$ by $Q(x) := \lim_{n\to\infty} \frac{f(a^n x)}{a^{2n}}$, namely, for each $t > 0$, and $x \in X$,

$$\lim_{n\to\infty} \nu(Q(x) - \frac{f(a^n x)}{a^{2n}})(t) = 1. \tag{46}$$

Let $x, y \in X$. Fix $t > 0$ and $0 < \varepsilon < 1$. Since $\{a^{-2n-2}\varphi(a^n x, 0)\}$ converges to zero, there is some $n_1 > n_0$ such that $t_0\varphi(a^n x, 0) < ta^{2n+2}$ for all $n \geq n_1$. Hence for each $n \geq n_1$, we have

$$\begin{aligned}
&\nu(Q(ax+by)+Q(ax-by)-2a^2Q(x)-2b^2Q(y))(t)\\
&\geq \Pi_{\mathcal{M}}\{\Pi_{\mathcal{M}}\{\nu(Q(ax+by)-\frac{f(a^{n+1}(ax+by))}{a^{2n+2}})(t),\\
&\nu(Q(ax-by)-\frac{f(a^{n+1}(ax-by))}{a^{2n+2}})(t)\},\Pi_{\mathcal{M}}\{\nu(2a^2Q(x)-a^{-2n-2}\cdot 2a^2f(a^{n+1}x))(t),\\
&\nu(2b^2Q(y)-a^{-2n-2}\cdot 2a^2f(a^{n+1}y))(t),\nu(f(a^{n+1}(ax+by))\\
&+f(a^{n+1}(ax-by)-2a^2f(a^{n+1}x)-2b^2f(a^{n+1}y))(a^{2n+2}t)\}\} \qquad (47)
\end{aligned}$$

The first four terms on the right-hand side of the above inequality tend to 1 as $n \to \infty$, and the fifth term is greater than

$$\nu(f(a^{n+1}(ax+by))+f(a^{n+1}(ax-by))-2a^2f(a^{n+1}x)-2b^2f(a^{n+1}y)(t_0\varphi(a^nx,0))$$

which is greater than or equal to $1-\varepsilon$. Thus

$$\nu(Q(ax+by)+Q(ax-by)-2a^2Q(x)-2b^2Q(y))(t) \geq 1-\varepsilon$$

for all $t>0$. It follows that $\nu(Q(ax+by)+Q(ax-by)-2a^2Q(x)-2b^2Q(y))(t)=1$ for all $t>0$. By (N1), we have $Q(ax+by)+Q(ax-by)-2a^2Q(x)-2b^2Q(y)=0$ for all $x \in X$. By Lemma 3.1, the mapping $Q : X \to \Upsilon$ is quadratic.

Next, let (37) holds for some positive δ and α. And we can put $m=0$ and $\alpha = 1-\varepsilon$ in (45) for all $x \in X$, we get

$$\nu(f(a^nx)-a^{2n}f(x))(\delta) \geq \alpha$$

for all positive integers $n \geq n_0$. Thus for large enough n, we have

$$\begin{aligned}
&\nu(f(x)-Q(x))(\delta a^{-2n-2}\varphi(a^nx,0)) \geq\\
&\Pi_{\mathcal{M}}\{\nu(f(x)-a^{-2n}f(a^nx)),\nu(a^{-2n}f(a^nx)-Q(x))\}(\delta a^{-2n-2}\varphi(a^nx,0)) \geq \alpha,
\end{aligned}$$

therefore

$$\nu(Q(x)-f(x))(\delta\tilde{\varphi}_n(x,0)) \geq \alpha.$$

The existence of uniform limit (39) immediately follows from the proof of the first part of Theorem 3.1. It remains to prove the uniqueness assertion. Let Q' be another quadratic mapping satisfying (2) and (39). Fix $c>0$. Given $\varepsilon>0$, by (39) for Q and Q', we can choose some t_0 such that

$$\nu(f(x)-Q(x))(t\tilde{\varphi}_n(x,0)) \geq 1-\varepsilon, \qquad \nu(f(x)-Q'(x))(t\tilde{\varphi}_n(x,0)) \geq 1-\varepsilon$$

for all $x \in X$ and $t \geq 2t_0$. Fix some $x \in X$ and find some integer n_0 such that

$$t_0a^{-2n}\varphi(a^nx,0) < c,$$

for all $n \geq n_0$. Thus we have

$$\begin{aligned}
\nu(Q(x)-Q'(x))(c) &\geq \Pi_{\mathcal{M}}\{\nu(a^{-2n}f(a^nx)-Q'(x)),\nu(Q(x)-a^{-2n}f(a^nx))\}(c)\\
&= \Pi_{\mathcal{M}}\{\nu(f(a^nx)-Q'(a^nx)),\nu(Q(a^nx)-f(a^nx))\}(a^{2n}c)\\
&\geq \Pi_{\mathcal{M}}\{\nu(f(a^nx)-Q'(a^nx)),\nu(Q(a^nx)-f(a^nx))\}(t_0\varphi(a^nx,0))\\
&\geq 1-\varepsilon.
\end{aligned}$$

It follows that $\nu(Q(x) - Q'(x))(c) = 1$ for all $c > 0$. Thus $Q(x) = Q'(x)$ for all $x \in X$. $\square$

Corollary 3.1. *Let X be a linear normed space and $(\Upsilon, \nu, \Pi_{\mathcal{M}})$ be a* Šerstnev *MPB space. Let $\theta \geq 0$ and let p be a real number with $0 \leq p < 2$ if $|a| > 1$. Suppose that $f : X \to \Upsilon$ is a mapping with $f(0) = 0$ such that*

$$\lim_{t\to\infty} \nu(f(ax+by) + f(ax-by) - 2a^2 f(x) - 2b^2 f(y))(t\theta(\|x\|^p + \|y\|^p)) = 1 \tag{48}$$

uniformly on $X \times X$. Then $Q(x) := \lim_{n\to\infty} a^{-2n} f(a^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to \Upsilon$ such that if for some $\delta > 0, \alpha > 0$

$$\nu(f(ax+by) + f(ax-by) - 2a^2 f(x) - 2b^2 f(y))(\delta\theta(\|x\|^p + \|y\|^p)) \geq \alpha \tag{49}$$

for all $x, y \in X$, then

$$\nu(Q(x) - f(x))(\frac{a^{n(p-2)}}{a^2}\delta\theta\|x\|^p) \geq \alpha \tag{50}$$

for all $x, y \in X$. Furthermore, the quadratic mapping $Q : X \to \Upsilon$ is the unique mapping such that

$$\lim_{t\to\infty} \nu(Q(x) - f(x))(\frac{a^{n(p-2)}}{a^2}t\theta\|x\|^p) = 1 \tag{51}$$

uniformly on X.

Proof. Define $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ and apply Theorem 3.1 to get the result. $\square$

We are ready to give our nonuniform version of the Hyers-Ulam-Rassias theorem for equation (2) in Šerstnev MPB space.

Theorem 3.2. *Let X be a linear space and $(Z, \omega, \Pi_{\mathcal{M}})$ be a* Šerstnev *MPN space. Let $\psi : X^2 \to Z$ be a function such that for some $0 < \alpha < a^2$*

$$\omega(\psi(ax, ay))(t) \geq \omega(\alpha\psi(x, y))(t) \tag{52}$$

for all $x, y \in X$ and $t > 0$. Let $(\Upsilon, \nu, \Pi_{\mathcal{M}})$ be a Š*erstnev MPB space and let $f : X \to \Upsilon$ be a quadratic mapping with $f(0) = 0$ such that*

$$\nu(f(ax+by) + f(ax-by) - 2a^2 f(x) - 2b^2 f(y))(t) \geq \omega(\psi(x, y))(t) \tag{53}$$

for all $x, y \in X$ and $t > 0$. Then there exists unique quadratic mapping $Q : X \to \Upsilon$ such that

$$\nu(f(x) - Q(x))(t) \geq \omega(\frac{1}{a^2}\psi(x, 0))(2t) \tag{54}$$

for all $x \in X$ and $t > 0$.

Proof. Putting $y = 0$ in (53), we obtain

$$\nu(2f(ax) - 2a^2 f(x))(t) \geq \omega(\psi(x, 0))(t) \tag{55}$$

for all $x \in X$ and $t > 0$. Using (52) and induction on n, one can verify that

$$\omega(\psi(a^n x, 0))(t) \geq \omega(\alpha^n \psi(x, 0))(t) \tag{56}$$

for all $x \in X$ and $t > 0$. It follows from (55) and (56) that

$$\nu(a^{-2n} f(a^n x) - a^{-2n+2} f(a^{n-1} x))((\frac{\alpha^n}{a^{2n}})t) \geq \omega((\frac{1}{\alpha})\psi(x, 0))(2t). \tag{57}$$

Thus for all $n \geq m \geq 0, x \in X$ and $t > 0$, we have

$$\begin{aligned}
&\nu(a^{-2n}f(a^n x) - a^{-2m}f(a^m x))((\frac{\alpha^{m+1}}{a^{2m+2}})t) \\
&= \nu(\sum_{k=m+1}^{n} a^{-2k}f(a^k x) - a^{-2k+2}f(a^{k-1}x))((\frac{\alpha^{m+1}}{a^{2m+2}})t) \\
&\geq \omega((\frac{1}{\alpha})\psi(x,0))(2t). \qquad (58)
\end{aligned}$$

So we get

$$\nu(a^{-2n}f(a^n x) - a^{-2m}f(a^m x))(t) \geq \omega((\frac{1}{\alpha})\psi(x,0))((\frac{a^{2m+2}}{\alpha^{m+1}})2t). \qquad (59)$$

Fix $x \in X$. Thanks to the fact that $\lim_{s\to\infty} \omega((\frac{1}{\alpha})\psi(x,0))(s) = 1$, we deduce that $\{\frac{f(a^n x)}{a^{2n}}\}$ is a Cauchy sequence in Υ. Since $(\Upsilon, \nu, \Pi_{\mathcal{M}})$ is complete, this sequence converges to some point $Q(x) \in \Upsilon$. Using (59) with $m = 0$, we obtain

$$\begin{aligned}
\nu(Q(x) - f(x))(t) &\geq \Pi_{\mathcal{M}}\{\nu(Q(x) - a^{-2n}f(a^n x)), \nu(a^{-2n}f(a^n x) - f(x))\}(t) \\
&\geq \Pi_{\mathcal{M}}\{\nu(Q(x) - a^{-2n}f(a^n x))(t), \omega(\frac{1}{a^2}\psi(x,0))(2t)\}. \qquad (60)
\end{aligned}$$

Hence

$$\begin{aligned}
\nu(Q(x) - f(x))(t) &\geq \Pi_{\mathcal{M}}\{\lim_{n\to\infty} \nu(Q(x) - a^{-2n}f(a^n x))(t), \omega(\frac{1}{a^2}\psi(x,0))(2t)\} \\
&= \omega(\frac{1}{a^2}\psi(x,0))(2t).
\end{aligned}$$

The rest of this proof can be proved in a similar fashion as in the proof of Theorems 2.2 and 3.1. □

4. **Acknowledgements**

The authors are grateful to the referee for his/her helpful comments and suggestions. This research work is supported by the National Natural Science Foundation of China (Grant Nos. 11401190) and BSQD12077.

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