# APPLICATION OF MELLIN TRANSFORMS IN DETERMINATION THE PROBABILITY DISTRIBUTION OF THE STOCHASTIC VOLATILITY MODELS

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*In this paper, an application of the Stochastic Mellin Transforms (SMTs) in determination of the probability distribution functions (PDFs) of the Stochastic Volatility (SV) models is proposed. In that aim, three basic SV time series with Gaussian innovations have been considered, and their PDFs were obtained by using the inverse SMT formula.*

**Keywords:** Mellin transforms, SV models, approximation, Gaussian probability distribution.
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## 1. Introduction

*Stochastic Volatility (SV) models* belong to the very important class of non-linear time series models, which are primarily using in econometrics. The first SV model introduced by Taylor [1] undergone the number of modifications and generalizations until nowadays [2]-[8]. Thanks to these modifications and generalizations, the commonly used models in the analysis of the financial time series dynamics are obtained [9]-[12]. However, the main problem in SV modelling consists in finding the probability distribution of some particular time series, which would completely describe their behaviour and dynamics in time [13]-[15]. Further, the basic stochastic characteristic of the SV models is the presence of two sources of indeterminacy. As a consequence, their probability distribution functions (PDFs) have no closed form. In order to overcome these difficulties, *the Stochastic Mellin Transforms (SMTs)* are proposed here.

According to their basic definition, the Mellin transforms represent the integral transforms which can be interpreted as the multiplicative, two-sided Laplace transforms. They are closely connected to the theory of Dirichlet series, and found the applications in the number theory [16], the asymptotic expansions of harmonic sums [17], the approximation of some class of convolution operators [18], theory of gamma function and other related special functions [19], wavelet analysis [20], etc. The first application of Mellin transforms in stochastic theory is related to the pioneer work of Epstein [21]. Recently, they have been mostly used for studying of the products of various kind random variables (RVs) [22], some continuous moment estimation procedures [23], as well as in the stochastic analysis of the options pricing [24]-[26].

Here, the Mellin transforms are applied in determination of the probability distribution functions (PDFs) of the Gaussian type SV models. For this purpose, the Stochastic Mellin Transform (SMT), as well as *the inverse SMT formula*, are introduced in following Section 2. Also, some basic properties of the

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SMTs are listed and considered, especially the SMTs of the RVs with the standard Gaussian probability distribution. Section 3 and 4 contain the main results, i.e. the effective procedures for computation the SMTs of the SV models' time series. Moreover, the application of inverse SMT formula in determination the PDFs of three basic SV time series is described and analyzed. Finally, some concluding remarks are given in Section 5.

## 2. Stochastic Mellin Transforms

Let $(\Omega,\mathscr{F},P)$ be some probability space and let $X:\Omega\to R$ be the RV with continuous PDF $f_X(x)$. Denote as $X^+=\max\{X,0\}$ and $X^-=\max\{-X,0\}$ the positive and negative part of the RV $X$, respectively. Then, *the Stochastic Mellin Transform* of the RV $X$ is

$$\widetilde{f}_X(s)=\int_0^{+\infty}x^s\mathrm{d}F_{X^+}(x)+\gamma\int_0^{+\infty}x^s\mathrm{d}F_{X^-}(x), \tag{1}$$

where $\gamma\in\mathbb{C}$ is a constant such that $\gamma^2=1$. In the special case, which be hereafter most often considered, when $X\geq 0$ almost surely (a.s.), the Eq. (1) becomes:

$$\widetilde{f}_X(s)=E(X^s)=\int_0^{+\infty}x^s f_X(x)\,\mathrm{d}x. \tag{2}$$

According to this, immediately follows:

**Lemma 2.1.** *For arbitrary, mutually independent RVs $X,Y\geq 0$ (a.s.) with the continuous PDFs $f_X(x)$, $f_Y(y)$, respectively, and each $\alpha\in\mathbb{R}$, the following equalities hold:*

$$\begin{aligned}
&\text{(i).} && \widetilde{f}_{\alpha X}(s)=E(\alpha^s X^s)=\alpha^s\widetilde{f}_X(s),\\
&\text{(ii).} && \widetilde{f}_{X^\alpha}(s)=E(X^{\alpha s})=\widetilde{f}_X(\alpha s),\\
&\text{(iii).} && \widetilde{f}_{XY}(s)=\widetilde{f}_X(s)\widetilde{f}_Y(s).
\end{aligned}$$

Also, it is well-known that the transform given by Eq. (2) exists on *the fundamental strip*: $\mathscr{D}_X=\langle a,b\rangle=\{s\in\mathbb{C}\,|\,a<\mathrm{Re}(s)<b\}$, where

$$f_X(x)=\begin{cases} O\left(x^{-(a+1)}\right), & x\to 0^+,\\ O\left(x^{-(b+1)}\right), & x\to+\infty.\end{cases} \tag{3}$$

As the function $f_X(x)$ is continuous and $\int_0^{+\infty}f_X(x)\,\mathrm{d}x=1$, integral in Eq. (2) obviously exists for any $s\in\mathbb{C}$ which satisfies inequality $\mathrm{Re}(s)>-1$. Thus, the fundamental strip $\mathscr{D}_X$ of arbitrary RV $X\geq 0$ contains the set $\mathscr{D}=\langle -1,+\infty\rangle$. On the other hand, according to Eq. (2) the so-called *inverse SMT formula* follows:

$$f_X(x)=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}x^{-(s+1)}\,\widetilde{f}_X(s)\,\mathrm{d}s, \tag{4}$$

where $x\geq 0$ and $c\in\langle a,b\rangle$ is an arbitrary point on which the integral above does not depend. Furthermore, the SMT $\widetilde{f}_X(s)$ is holomorphic on $\langle a,b\rangle$ and by Cauchy's theorem, the path of integration of integral in Eq. (4) can be translated inside, without affecting the result of the integration.

**Example 2.1.** Consider the RV $Z$ with the Gaussian distribution $\mathscr{N}(0,1)$. Using Eq. (1) with $\gamma=1$, as well as the symmetry properties of this distribution, its SMT can be obtained as:

$$\widetilde{f}_Z(s)=E\left(|Z|^s\right)=\sqrt{\frac{2}{\pi}}\int_0^{+\infty}z^s\mathrm{e}^{-z^2/2}\mathrm{d}z=\frac{2^{s/2}}{\sqrt{\pi}}\Gamma\left(\frac{s+1}{2}\right). \tag{5}$$

Beside to this, the following formula is valid:

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} = \begin{cases} O\left(z^0\right), & z \to 0^+, \\ O\left(z^{-\infty}\right), & z \to +\infty, \end{cases}$$

and according to Eq. (3) it follows that the Eq. (5) holds for arbitrary $s \in \mathbb{C}$ which satisfies the condition $\mathrm{Re}(s) > -1$. Thus, the fundamental strip for the SMT $\widetilde{f}_Z(z)$ is $\mathscr{D}_Z = \langle -1, +\infty \rangle$. Finally, notice that when $s = 2n$, $n \in \mathbb{N}$, Eq. (5) implies the even moments of the RV $Z$ (the odd moments are equal zero):

$$\widetilde{f}_Z(2n) = E(Z^{2n}) = \frac{2^n}{\sqrt{\pi}} \Gamma\left(\frac{2n+1}{2}\right) = \frac{2^n}{\sqrt{\pi}} \prod_{k=1}^{n} \frac{2k-1}{2} \Gamma\left(\frac{1}{2}\right) = (2n-1)!!.$$

These facts will be useful below, where the procedure for determining the SMTs of the stochastic volatility models will be described in detail.

## 3. SMTs of the Stochastic Volatility Models

In general, the series $(X_t)$, when $t \in \mathbb{Z}$, represents *the Stochastic Volatility (SV) model* if it is defined by the following equations:

$$\begin{cases} X_t &= V_t^{1/2} \varepsilon_t \\ V_t &= \sigma^2 \exp(\Delta_t) \\ \Delta_t &= \alpha \Delta_{t-1} + \delta \eta_t. \end{cases} \tag{6}$$

Here, $\alpha$ is the autoregressive parameter which satisfies the nontriviality and stationarity condition $0 < |\alpha| < 1$, while $\sigma, \delta > 0$ are dispersion parameters. Further, $(\varepsilon_t)$ and $(\eta_t)$ are series of independent identically distributed (i.i.d.) RVs with, an usually, Gaussian distribution $\mathscr{N}(0,1)$. In that way, they represent two different sources of the uncertainty (popularly named "noise-series") in the SV model. On the other hand, $(X_t)$ is *the basic SV series*, i.e., it represents the realized values of some real-based time series. The second series $(V_t)$ is usually named *the volatility*, and it is a well-known measure of uncertainty in the fluctuations of the series $(X_t)$. Finally, the series $(\Delta_t)$ defined by the third, recurrence relation in Eq. (6), is *an autoregressive (AR) sequence* of RVs and it represents the linear component of SV model. Realizations of these series of length $T = 10000$, along with their empirical PDFs (shown with histograms), where the parameters values are $\alpha = 0.5$ and $\sigma = \delta = 1$, are plotted in Fig. 1. Now, in order to determine theoretical PDFs of the SV model's time series, using the aforementioned facts, firstly we will determine their SMTs in the following way.

**Theorem 3.1.** *For the SV model defined by the Eqs. (6), i.e. for the series $(\Delta_t)$, $(V_t)$ and $(X_t)$, the appropriate SMTs are, respectively:*

$$(i).\quad \widetilde{f}_\Delta(s) = \frac{1}{\sqrt{\pi}} \left(\frac{2\delta^2}{1-\alpha^2}\right)^{s/2} \Gamma\left(\frac{s+1}{2}\right);$$

$$(ii).\quad \widetilde{f}_V(s) = \sigma^{2s} \exp\left(\frac{\delta^2 s^2}{2(1-\alpha^2)}\right);$$

$$(iii).\quad \widetilde{f}_X(s) = \frac{\left(\sqrt{2}\sigma\right)^s}{\sqrt{\pi}} \Gamma\left(\frac{s+1}{2}\right) \exp\left(\frac{\delta^2 s^2}{8(1-\alpha^2)}\right).$$

*In addition, the fundamental strip for all of these SMTs is $\mathscr{D} = \langle -1, +\infty \rangle$.*

*Proof.* $(i)$ Notice that $(\Delta_t)$ is ergodic and stationary series of the RVs, with $E(\Delta_t) = 0$. Moreover, for arbitrary $t, k \in \mathbb{Z}$ the RVs $(\Delta_t)$ can be expressed on $\Delta_{t-j},\ j = 1, \ldots, k$ as

$$\Delta_t = \alpha^k \Delta_{t-k} + \delta \sum_{j=0}^{k-1} \alpha^j \eta_{t-j}. \tag{7}$$

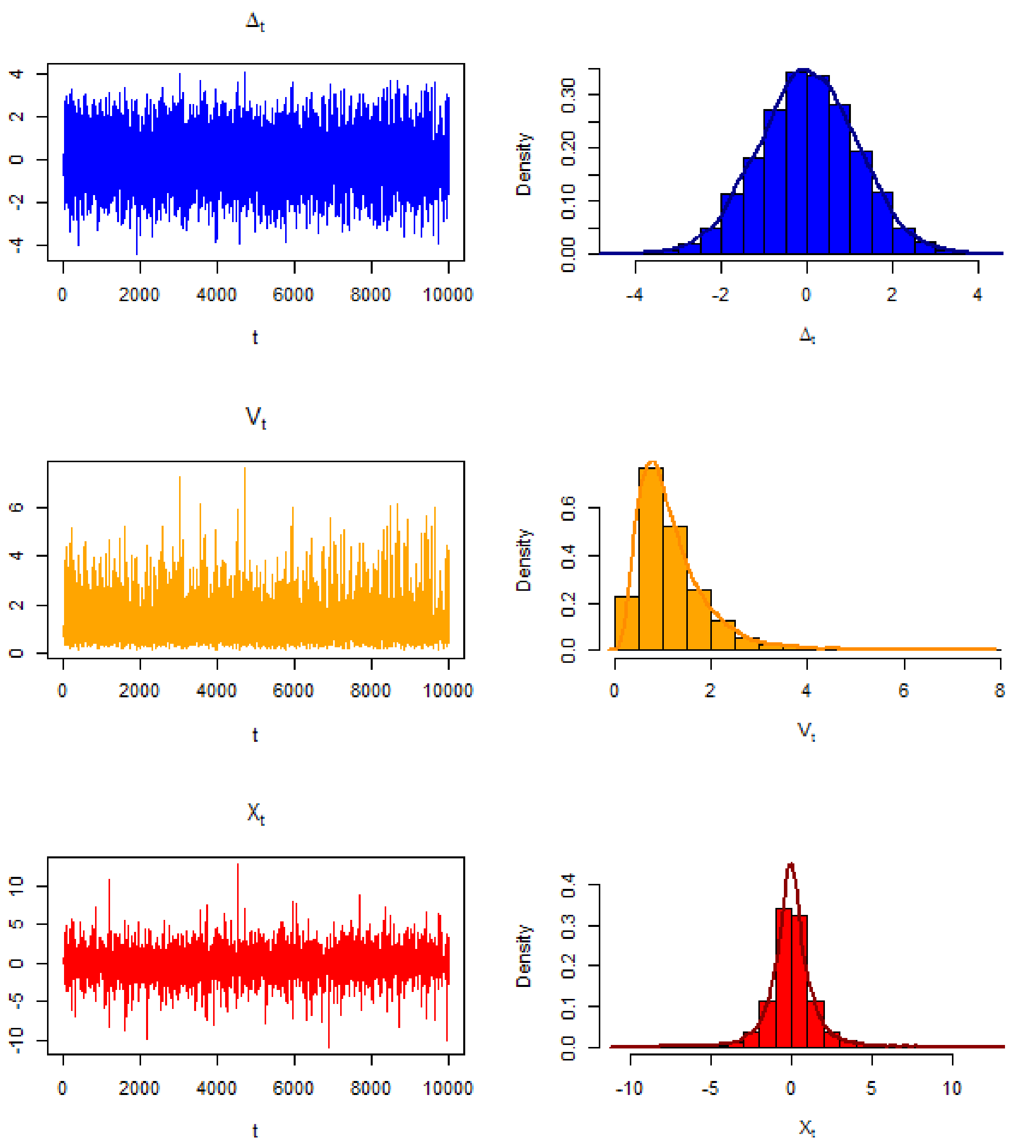


FIGURE 1. Diagrams left: Realizations of length $T = 10000$ of the series $(\Delta_t)$, $(V_t)$ and $(X_t)$, respectively. Diagrams right: Empirical PDFs of the realized SV-series.

From Eq. (7), when $|\alpha| < 1$, it follows:

$$\lim_{k\to+\infty} E\left[\Delta_t - \delta\sum_{j=0}^{k-1}\alpha^j\eta_{t-j}\right]^2 = \lim_{k\to+\infty}\alpha^{2k}\,\mathrm{Var}(\Delta_{t-k}) = 0,$$

i.e. the sum in Eq. (7) converges in mean-square, when $k \to +\infty$, to the RV $\Delta_t$. Similarly, Kolmogorov's large number low implies the almost surely convergence of this sum. Thus, for any $t \in \mathbb{Z}$ the following equality holds:

$$\Delta_t \overset{d}{=} \delta \sum_{j=0}^{\infty} \alpha^j \eta_{t-j},$$

i.e. the RVs $(\Delta_t)$ have the Gaussian distribution $\mathscr{N}(0, \nu^2)$, with the variance:

$$\nu^2 := \mathrm{Var}(\Delta_t) = \delta^2 \sum_{j=0}^{\infty} \alpha^{2j} \mathrm{Var}(\eta_j) = \frac{\delta^2}{1-\alpha^2}. \tag{8}$$

According to this, for arbitrary RV $Z$ with the Gaussian distribution $\mathscr{N}(0,1)$, the equality $\Delta_t \overset{d}{=} \nu Z$ holds. From this, using the property (i) from Lemma 2.1, we obtain the SMT of RVs $(\Delta_t)$ as $\widetilde{f}_\Delta(s) = \nu^s \widetilde{f}_Z(s)$. Then, by substituting Eqs. (5) and (8) in the last equality, the statement of this part of theorem follows.

$(ii)$ Using Eq. (2), as well as the definition of the volatility given by the second equality in Eqs. (6), we obtain the appropriate SMT of the RVs $(V_t)$ as:

$$\widetilde{f}_V(s) = E\left[\left(\sigma^2 \mathrm{e}^{\Delta_t}\right)^s\right] = \sigma^{2s} E\left(\mathrm{e}^{s\Delta_t}\right) = \frac{\sigma^{2s}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{\nu s z - z^2/2} \mathrm{d}z = \frac{\sigma^{2s}\, \mathrm{e}^{\nu^2 s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-u^2/2} \mathrm{d}u$$

$$= \sigma^{2s} \exp\left(\frac{\nu^2 s^2}{2}\right),$$

where $u := z - s\nu$. Substitution of $\nu^2$ (8) in the last expression ends the proof of this part of the theorem.

$(iii)$ According to the first equality in Eqs. (6), the previously obtained expression for SMT of the series $(V_t)$, and the properties (ii) and (iii) from Lemma 2.1, the SMT of the series $(X_t)$ immediately follows from equalities:

$$\widetilde{f}_X(s) = E\left(V_t^{s/2}\right) \widetilde{f}_\varepsilon(s) = \widetilde{f}_V\left(\frac{s}{2}\right) \frac{2^{s/2}}{\sqrt{\pi}} \Gamma\left(\frac{s+1}{2}\right),$$

where the SMT $\widetilde{f}_\varepsilon(s)$ of the series $(\varepsilon_t)$ is the same as one in Eq. (5). □

**Remark 3.1.** Similarly to the Gaussian RVs, according to the SMT of the basic SV series $(X_t)$, the moments of even order of these RVs can be easily obtained (the moments of odd order are equal zero):

$$E\left(X_t^{2n}\right) = \widetilde{f}_X(2n) = \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \Gamma\left(\frac{2n+1}{2}\right) \exp\left(\frac{n^2\delta^2}{2(1-\alpha^2)}\right) = (2n-1)!!\, \sigma^{2n} \exp\left(\frac{n^2\delta^2}{2(1-\alpha^2)}\right).$$

Moreover, when $s > -1$, the SMT $\widetilde{f}_X(s)$ represents the $s$-th moments of the RVs $(X_t)$.

## 4. Determination the PDFs of the Stochastic Volatility Models

In this section, application of the inverse Mellin formula, given by Eq. (4), in the determination of the PDFs of the SV model's time series is presented. The following statement gives the main results, i.e. explicit expressions for the PDFs of the AR series $(\Delta_t)$, the volatility $(V_t)$, as well as the basic series $(X_t)$.

**Theorem 4.1.** *The PDFs of the series $(\Delta_t)$, $(V_t)$ and $(X_t)$, defined by the Eqs. (6) are, respectively:*

$$(i).\quad f_\Delta(x) = \sqrt{\frac{1-\alpha^2}{2\pi\delta^2}} \exp\left(-\frac{(1-\alpha^2)x^2}{2\delta^2}\right), \quad x \in \mathbb{R};$$

$$(ii).\quad f_V(x) = \sqrt{\frac{1-\alpha^2}{2\pi\delta^2 x^2}} \exp\left(-\frac{(1-\alpha^2)\ln^2\left(x/\sigma^2\right)}{2\delta^2}\right), \quad x > 0;$$

$$(iii).\quad f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{x^2}{2\sigma^2}\right)^k \exp\left(\frac{\delta^2(2k+1)^2}{8(1-\alpha^2)}\right), \quad x \in \mathbb{R}.$$

*Proof.* $(i)$ The first equality directly follows from the fact that RVs $(\Delta_t)$ have a Gaussian distribution $\mathscr{N}(0,\nu^2)$, with $\nu^2=\delta^2/(1-\alpha^2)$.

$(ii)$ By applying the inverse Mellin formula, i.e. Eq. (4) on the SMT $\widetilde{f}_V(s)$, we obtain:

$$\begin{aligned}
f_V(x) &= \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} x^{-s-1}\widetilde{f}_V(s)\mathrm{d}s = \frac{1}{2\pi \mathrm{i}x}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\left(\frac{\sigma^2}{x}\right)^s \exp\left(\frac{\nu^2 s^2}{2}\right)\mathrm{d}s \\
&= \frac{1}{2\pi \mathrm{i}x}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\exp\left(\frac{\nu^2 s^2}{2} - s\ln\left(\frac{x}{\sigma^2}\right)\right)\mathrm{d}s \\
&= \frac{\exp\left(-\ln^2\left(x/\sigma^2\right)/2\nu^2\right)}{2\pi\nu x}\int_{-\infty}^{+\infty}\exp\left(\frac{-u^2}{2}\right)\mathrm{d}u \\
&= \frac{x^{-1}}{\sqrt{2\pi\nu^2}}\exp\left(-\frac{\ln^2\left(x/\sigma^2\right)}{2\nu^2}\right), \quad x>0.
\end{aligned} \tag{9}$$

Here, we have taken $c=0\in\mathscr{D}=\langle -1,+\infty\rangle$, as well as $u=\mathrm{i}\left(\nu s-\nu^{-1}\ln(x/\sigma^2)\right)$ and $\nu=\delta/(1-\alpha^2)^{1/2}$. Obviously, the last expression in Eqs. (9) gives the statement of this part of theorem.

$(iii)$ According to the symmetry properties of the PDF $f_X(x)=f_X(-x)$, it is sufficient to consider the case when $x>0$. Thus, $f_X(x)$ can be written as the inverse Mellin transform of its appropriate SMT $\widetilde{f}_X(s)$ in following way:

$$\begin{aligned}
f_X(x) &= \frac{1}{2}\left(\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} x^{-(s+1)}\widetilde{f}_X(s)\,\mathrm{d}s\right) \\
&= \frac{1}{4\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} x^{-(s+1)}\frac{\left(\sqrt{2}\sigma\right)^s}{\sqrt{\pi}}\Gamma\left(\frac{s+1}{2}\right)\exp\left(\frac{\nu^2 s^2}{8}\right)\mathrm{d}s \\
&= \frac{1}{4\pi^{3/2}}\int_{-\infty}^{+\infty} x^{-(\mathrm{i}\xi+1)}\left(\sqrt{2}\sigma\right)^{\mathrm{i}\xi}\Gamma\left(\frac{\mathrm{i}\xi+1}{2}\right)\exp\left(-\frac{\nu^2\xi^2}{8}\right)\mathrm{d}\xi \\
&= \frac{1}{4\pi^{3/2}x}\int_{-\infty}^{+\infty}\exp\left(\mathrm{i}\xi\ln\left(\sqrt{2}\sigma/x\right)\right)g(\xi)\mathrm{d}\xi,
\end{aligned} \tag{10}$$

where $\xi=\mathrm{i}s$ and $g(\xi):=\Gamma((\mathrm{i}\xi+1)/2)\exp\left(-\nu^2\xi^2/8\right)$. According to the well-known properties of Gamma function $\Gamma(z)$, $z\in\mathbb{C}$, it follows that function $g(\xi)$ has the simple poles in the upper complex half-plane $G=\left\{\xi\in\mathbb{C}\,\middle|\,\mathrm{Im}(\xi)>0\right\}$, when $(\mathrm{i}\xi+1)/2=-k$, i.e. at the points $\xi_k=(2k+1)\mathrm{i}$, where $k=0,1,2,\dots$ (see Fig. 2). Then, the residues of $g(\xi)$ in these points are:

$$\begin{aligned}
\mathrm{Res}\left[g(\xi),\,\xi_k\right] &:= \lim_{\xi\to\xi_k}\left[(\xi-\xi_k)\,g(\xi)\right] = \lim_{\xi\to\xi_k}\left[(\xi-\xi_k)\,\Gamma\left(\frac{\mathrm{i}\xi+1}{2}\right)\exp\left(-\frac{\nu^2\xi^2}{8}\right)\right] \\
&= -2\mathrm{i}\,\mathrm{Res}\left[\Gamma(z),\,-k\right]\exp\left(-\frac{\nu^2\xi_k^2}{8}\right) = -2\mathrm{i}\,\frac{(-1)^k}{k!}\exp\left(\frac{\nu^2(2k+1)^2}{8}\right).
\end{aligned} \tag{11}$$

On the other hand, $g(\xi)$ is a holomorphic function on the set $G'=G\setminus\{\xi_k\,|\,k=0,1,2,\dots\}$ and it has an analytic continuation to the whole half-plane $G$. Furthermore, for any $\xi\in G'$ there exist $N\in\{0,1,2\dots\}$ such that $\mathrm{Re}(\xi)\neq 0$ and $2N-1\le\mathrm{Im}(\xi)<2N+1$, or $\mathrm{Re}(\xi)=0$ and $2N-1<\mathrm{Im}(\xi)<2N+1$

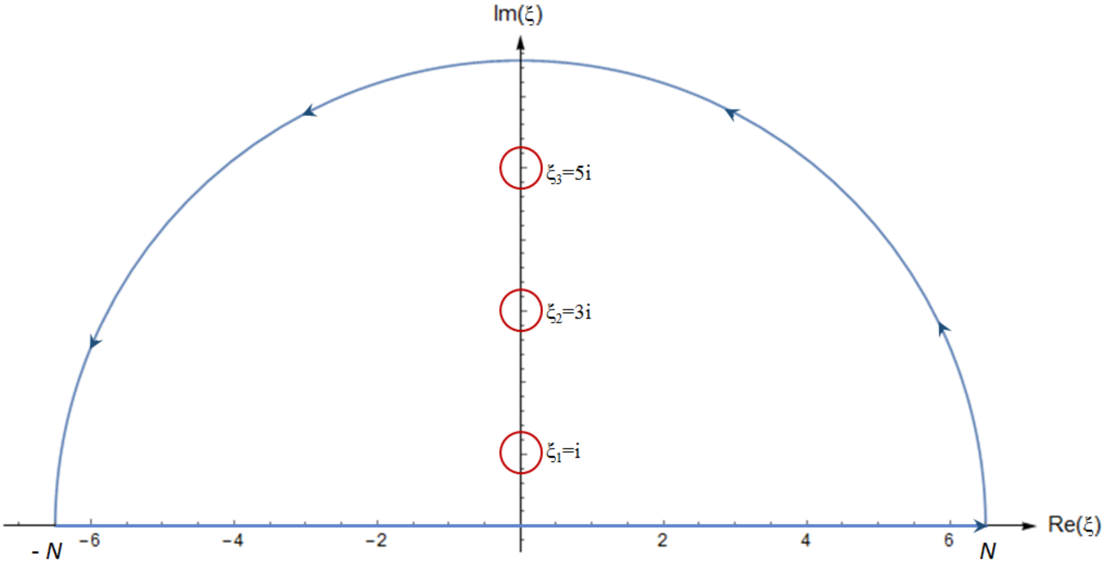


FIGURE 2. Contour of integration of the function $\exp\left(\mathrm{i}\xi\ln(\sqrt{2}\sigma/x)\right)g(\xi)$.

hold. Thus, on the set $G'$ the following equality is valid:

$$\Gamma\left(\frac{\mathrm{i}\xi+1}{2}\right)=\frac{\Gamma\left(\frac{\mathrm{i}\xi+1}{2}+N\right)}{\frac{\mathrm{i}\xi+1}{2}\left(\frac{\mathrm{i}\xi+1}{2}+1\right)\cdots\left(\frac{\mathrm{i}\xi+1}{2}+N-1\right)},$$

where $\mathrm{Re}\left((\mathrm{i}\xi+1)/2+N\right)=(2N+1-\mathrm{Im}(\xi))/2>0$ and $\mathrm{Re}\left((\mathrm{i}\xi+1)/2+k\right)=(2k+1-\mathrm{Im}(\xi))/2\le 0$, $k=0,1,\ldots,N-1$. In that way, when $N\longrightarrow+\infty$ (i.e. $|\xi|\to+\infty$), the function $g(\xi)$ satisfies:

$$\begin{aligned}
|g(\xi)| &= \frac{\left|\int_0^{+\infty} z^{(\mathrm{i}\xi-1)/2+N}\mathrm{e}^{-z}\,\mathrm{d}z\right|}{\left|\frac{\mathrm{i}\xi+1}{2}\right|\left|\frac{\mathrm{i}\xi+1}{2}+1\right|\cdots\left|\frac{\mathrm{i}\xi+1}{2}+N-1\right|\exp\left(v^2\xi^2/8\right)}\\
&\le \frac{\int_0^{+\infty} z^{N-1/2}\left|\mathrm{e}^{\frac{\mathrm{i}\xi}{2}\ln z}\right|\mathrm{e}^{-z}\,\mathrm{d}z}{\left|\frac{\mathrm{i}\xi+1}{2}\right|\left|\frac{\mathrm{i}\xi+1}{2}+1\right|\cdots\left|\frac{\mathrm{i}\xi+1}{2}+N-1\right|\exp\left(v^2\xi^2/8\right)}\\
&\le \frac{\int_0^{+\infty} z^{N-1/2}\exp\left(-\frac{1}{2}(2N-1)\ln z-z\right)\mathrm{d}z}{\left|\frac{\mathrm{i}\xi+1}{2}\right|\left|\frac{\mathrm{i}\xi+1}{2}+1\right|\cdots\left|\frac{\mathrm{i}\xi+1}{2}+N-1\right|\exp\left(v^2\xi^2/8\right)}\\
&\le \frac{1}{\left|\frac{\mathrm{i}\xi+1}{2}\right|\left|\frac{\mathrm{i}\xi+1}{2}+1\right|\cdots\left|\frac{\mathrm{i}\xi+1}{2}+N-1\right|\exp\left(v^2\xi^2/8\right)}\longrightarrow 0.
\end{aligned}$$

By applying Jordan's lemma (see, for instance [27]), as well as Eqs. (11), the last integral in Eqs. (10) becomes:

$$\begin{aligned} f_X(x) &= \frac{1}{4\pi^{3/2}x} \lim_{N\to+\infty} \int_{-N}^{N} \exp\Big(\mathrm{i}\xi \ln(\sqrt{2}\sigma/x)\Big)\, g(\xi)\, \mathrm{d}\xi \\ &= \frac{1}{4\pi^{3/2}x} \lim_{N\to+\infty} \left[2\pi\mathrm{i} \sum_{k=0}^{N} \exp\Big(\mathrm{i}\xi_k \ln\Big(\sqrt{2}\sigma/x\Big)\Big) \operatorname{Res}\big[g(\xi),\ \xi_k\big]\right] \\ &= \frac{\mathrm{i}}{2\sqrt{\pi}x} \lim_{N\to+\infty} \sum_{k=0}^{N} -2\mathrm{i}\,\frac{(-1)^k}{k!} \exp\left(-(2k+1)\ln\left(\frac{\sqrt{2}\sigma}{x}\right)\right) \exp\left(\frac{\nu^2(2k+1)^2}{8}\right) \\ &= \frac{1}{\sqrt{\pi}} \lim_{N\to+\infty} \sum_{k=0}^{N} \frac{(-1)^k}{k!} x^{2k} \left(\sqrt{2}\sigma\right)^{-(2k+1)} \exp\left(\frac{\nu^2(2k+1)^2}{8}\right) \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{x^2}{2\sigma^2}\right)^k \exp\left(\frac{\nu^2(2k+1)^2}{8}\right). \end{aligned} \tag{12}$$

Finally, substitution $\nu^2 = \delta^2/(1-\alpha^2)$ in the last of Eqs. (12) ends the proof of theorem. □

**Remark 4.1.** The PDF $f_X(x)$ of the basic SV time series $(X_t)$, given by the last of Eqs. (12), represents the so-called alternating series. If we denote the absolute value of the $k$-th term as

$$a_k(x) := \frac{1}{k!}\left(\frac{x^2}{2\sigma^2}\right)^k \exp\left(\frac{\delta^2(2k+1)^2}{8(1-\alpha^2)}\right), \quad k = 0,1,2,\dots$$

then, according to Leibniz criterion, the sufficient condition of (conditional) convergence of this series is that the sequence $(a_k)$ converges to zero monotonically. After some computation, it can be easily obtained:

$$a_{k-1}(x) - a_k(x) = \frac{1}{k!}\left(\frac{x^2}{2\sigma^2}\right)^{k-1} \exp\left(\frac{\delta^2(2k-1)^2}{8(1-\alpha^2)}\right)\left[k - \frac{x^2}{2\sigma^2}\exp\left(\frac{k\delta^2}{1-\alpha^2}\right)\right], \tag{13}$$

so that $a_{k-1}(x) - a_k(x) > 0$ holds if and only if is $|x| < R_k$, where

$$R_k := \sigma\sqrt{2k}\exp\left(-\frac{k\delta^2}{2(1-\alpha^2)}\right), \quad k = 1,2,\dots \tag{14}$$

Thus, for a given $k = k_0 \in \mathbb{N}$, the PDF $f_X(x)$ can be approximated on $x \in \left(-R_{k_0}, R_{k_0}\right)$ with the partial sum:

$$f_{k_0}(x) := \frac{1}{\sqrt{2\pi\sigma^2}} \sum_{j=0}^{k_0} \frac{1}{j!}\left(-\frac{x^2}{2\sigma^2}\right)^j \exp\left(\frac{\nu^2(2j+1)^2}{8}\right). \ \square \tag{15}$$

In Fig. 3 are shown 3-dimensional plots of the modulus of the SMTs of the autoregressive Gaussian series $(\Delta_t)$, the volatility $(V_t)$ and the series $(X_t)$, as well as plots of the appropriate PDFs of these series. Notice that, according to Eqs. (13)-(14), the 'small' parameter $\delta > 0$ enables that the approximation given by Eq. (15) is valid for 'large' $k_0 \in \mathbb{N}$. Moreover, when $\delta \to 0^+$, the last of Eqs. (12) gives:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{x^2}{2\sigma^2}\right)^k = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right),$$

and, in this case, the series $(X_t)$ has the standard Gaussian distribution $\mathcal{N}(0,\sigma^2)$.

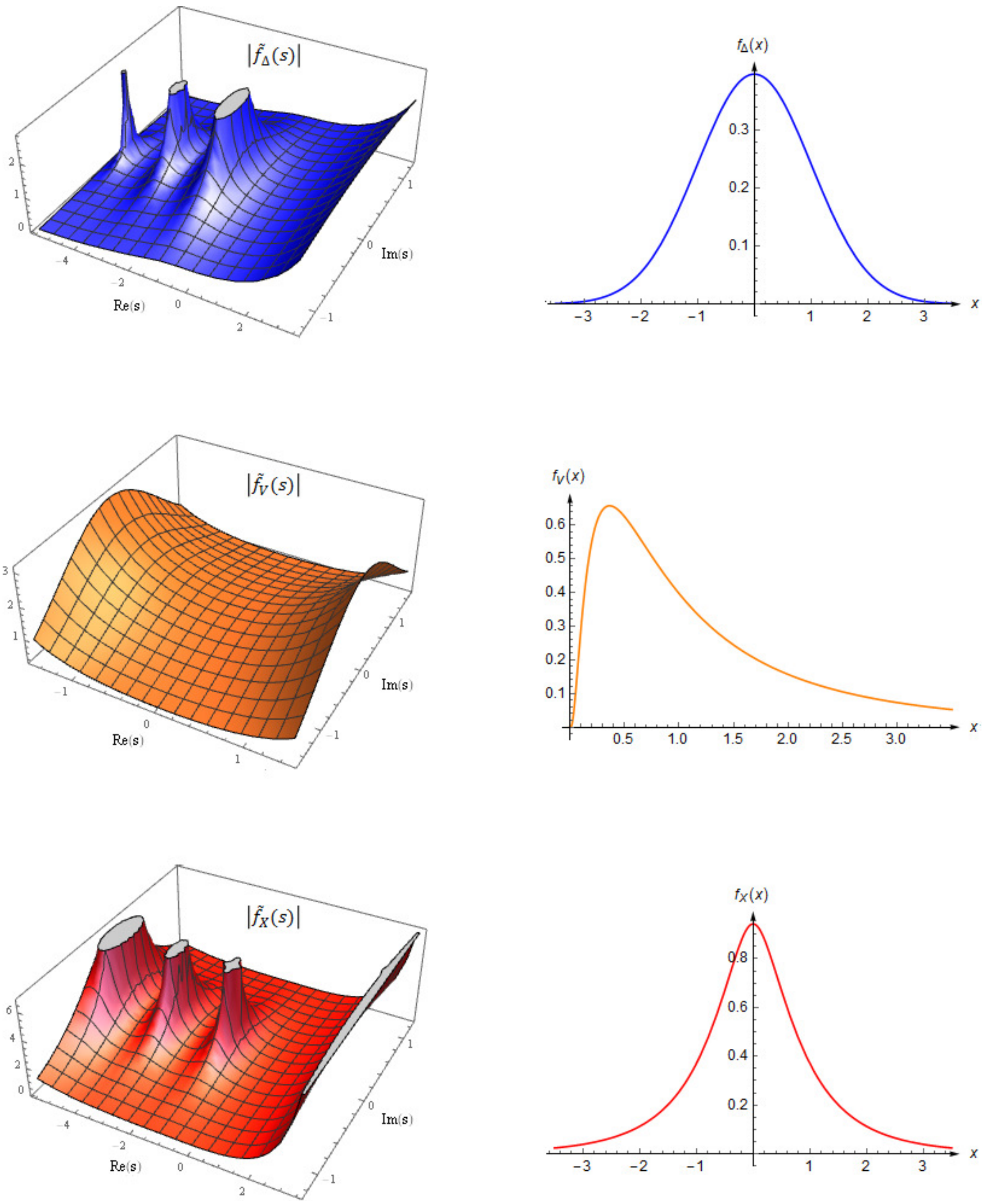


FIGURE 3. Diagrams left: 3D plots of the modulus of the SMTs of the series $(\Delta_t)$, $(V_t)$ and $(X_t)$. Diagrams right: Plots of the PDFs of the series $(\Delta_t)$, $(V_t)$ and $(X_t)$. (The parameters values are: $\alpha = 0.5$ and $\sigma = \delta = 1$.)

## 5. Conclusion

The procedure of determining the probability distribution functions (PDFs) of the time series in stochastic volatility (SV) model is described. For the purpose of the PDFs determination, the stochastic Mellin transformations (SMTs) are proposed. The PDFs of the SV models time series with Gaussian distributed innovations were obtained in accordance to inverse SMT formula. Additionally, the described treatment can be also applied in a similar way to determination of the probability distribution of the SV models with some different, non-Gaussian innovations (such as, for example, Student $t$-distribution).

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