ON $\mathfrak{N}$-PRIME IDEALS

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In this article, we introduce an intermediate classes of ideals between prime and quasi primary ideals, denoted by $\mathfrak{N}$-prime, and we focus on some properties of $\mathfrak{N}$-prime ideals. Moreover, we defined a topology on the set of all $\mathfrak{N}$-prime ideals such that we examine the topological concepts, irreducibility, connectedness, and seperation axioms.

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1. Introduction

Throughout this study, all rings will be commutative with $1 \neq 0$. Let R denote such a ring. If P is an ideal of R, then the radical of P, $\sqrt{P}$, is defined to be

$$\sqrt{P} := \Big\{a \in R : a^n \in P \text{ for some n} \in \mathbb{N}\Big\}.$$

We denote the nilradical of R by $\mathfrak{N}(R)$ instead of $\sqrt{0}$. The concept of prime ideal has a significant role in the theory of commutative algebra and algebraic geometry. The properties of prime ideals in a special ring has been obtained in different articles, see [3-4,6,9]. Recall from [2], a prime ideal P of R is a proper ideal if $ab \in P$ implies $a \in P$ or $b \in P$ for each $a, b \in R$. A proper ideal Q of R is called primary if whenever $ab \in Q$, then $a \in Q$ or $b \in \sqrt{Q}$, equivalently $a \in \sqrt{Q}$ or $b \in Q$, [10]. Also a quasi primary ideal Q of R is defined as a proper ideal whose radical is prime [7].

The main focus in this study (especially in Chapter 2) is to present an intermediate classes of ideals between prime and quasi primary ideals, and to examine its properties, called $\mathfrak{N}$-prime ideals. We will define P is a $\mathfrak{N}$-prime ideal of R to be a proper ideal P satisfying the condition $ab \in P$ implies either $a \in P + \mathfrak{N}(R)$ or $b \in P + \mathfrak{N}(R)$. Among many results in Chapter 2, we give (in Corollary 2.2) a number of results characterizing the $\mathfrak{N}$-prime ideals of a given ring R. Also, we determine all $\mathfrak{N}$-prime ideals of cartesian products of rings. Recall the crucial theorem of prime avodiance lemma. Suppose that $I \subseteq \bigcup_{i=1}^{n} P_i$ is a covering of prime ideals,

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all P_i's are prime ideals, then at least one of them contains I. We examine the lemma for $\mathfrak{N}$-prime ideals. Moreover, we study the $\mathfrak{N}$-prime ideals of fractional ring $S^{-1}R$. And we characterize the $\mathfrak{N}$-prime homogeneous ideals of idealization of a unital R-module M. A ring (not necessarily commutative) R is called a UN-ring if every nonunit element of R is a product of a unit and nilpotent element, [5]. We characterize all UN-rings by means of $\mathfrak{N}$-prime ideals. We support each results with examples.

In Chapter 3, we construct a topology on $\mathfrak{Nspec}(\mathfrak{R})$, where $\mathfrak{Nspec}(\mathfrak{R})$ denotes the set of all $\mathfrak{N}$-prime ideals of R while $Spec(R)$ denotes the set of all prime ideals of R. We show that the topological spaces of $Spec(R)$ and $\mathfrak{Nspec}(\mathfrak{R})$ are different. Moreover, we obtained some topological properties of $\mathfrak{Nspec}(\mathfrak{R})$, and we support the results with some examples.

2. $\mathfrak{N}$-prime Ideals in Commutative Rings

Definition 2.1. *A proper ideal P of R is called a $\mathfrak{N}$-prime ideal if $ab \in P$, for each $a, b \in R$, then either $a \in P + \mathfrak{N}(R)$ or $b \in P + \mathfrak{N}(R)$.*

Example 2.1. *(i) In a reduced ring; prime and $\mathfrak{N}$-prime ideals coincide. In particular, in any domain or von Neumann regular ring, all $\mathfrak{N}$-prime ideals are exactly prime ideals.*

(ii) Let (R, M) be a quasi-local with nil maximal ideal, i.e, $M = \mathfrak{N}(R)$. If P is a proper ideal of R and $ab \in P$ for $a, b \in R$, then $a \in P + \mathfrak{N}(R)$ or $b \in P + \mathfrak{N}(R)$. Thus every proper ideal is $\mathfrak{N}$-prime ideal in a quasi local ring with nil maximal.

(iii) Consider the quotient ring

$$R = F[X, Y]/\langle X^2\rangle,$$

where F is a field, and the ideal

$$P = \langle X^2, XY, Y^2\rangle/\langle X^2\rangle.$$

Note that $\mathfrak{N}(R) = \langle x\rangle$ and $P + \mathfrak{N}(R) = \langle x, y^2\rangle$, where

$$x = X + \langle X^2\rangle \text{ and } y = Y + \langle X^2\rangle.$$

Since $y^2 \in P$ and $y \notin P + \mathfrak{N}(R)$, P is not a $\mathfrak{N}$-prime ideal of R.

Fact 2.1. *(i) Assume that $P + \mathfrak{N}(R)$ is a prime ideal of R. Then it is easily seen that P is a $\mathfrak{N}$-prime ideal: if $ab \in P \subseteq P + \mathfrak{N}(R)$, then it follows that $a \in P + \mathfrak{N}(R)$ or $b \in P + \mathfrak{N}(R)$.*

(ii) Since prime ideals contain the nilradical, every prime ideal is also a $\mathfrak{N}$-prime ideal. However, the converse is not hold. For instance, consider the ring $\mathbb{Z}_{36}$ and the ideal $P = \langle\overline{4}\rangle$. It is clear that P is not a prime ideal. In addition, $P + \mathfrak{N}(\mathbb{Z}_{36}) = \langle\overline{2}\rangle$ is prime, then by (i), P is a $\mathfrak{N}$-prime ideal of $\mathbb{Z}_{36}$.

(iii) If P is a $\mathfrak{N}$-prime ideal of R and $\mathfrak{N}(R) \subseteq P$, then P is a prime ideal of R.

The following explicit result is easily obtained from Fact 2.1.

Corollary 2.1. *A proper ideal P of a ring R is prime if and only if P is $\mathfrak{N}$-prime and $\mathfrak{N}(R) \subseteq P$.*

The following examples show the differences between primary ideals and $\mathfrak{N}$-prime ideals.

Example 2.2. *(i) Assume that R is a PID and $0 \neq p$ is an irreducible element. It is obvious that $\langle p^n \rangle$ is a primary ideal for $n > 1$ but it is not a $\mathfrak{N}$-prime ideal.*

(ii) Let $R = \mathbb{Z}_8[X,Y]$ and

$$\psi : \mathbb{Z}_8[X,Y] \to \mathbb{Z}_2[X,Y]$$

be homomorphism defined by

$$\psi(g_0(X)+g_1(X)Y+g_2(X)Y^2+...+g_n(X)Y^n) = \overline{g_0(X)}+\overline{g_1(X)}Y+\overline{g_2(X)}Y^2+...+\overline{g_n(X)}Y^n,$$

where $\overline{g_i(X)}$ is a polynomial obtained by taking the coefficient of $g_i(X)$ in modulo 2. Note that

$$Ker(\psi) = \mathfrak{N}(R) = \overline{2}\mathbb{Z}_8[X,Y]$$

and ψ is an epimorphism. Thus

$$\mathbb{Z}_8[X,Y]/\mathfrak{N}(R) \cong \mathbb{Z}_2[X,Y]$$

is an integral domain, so that $\mathfrak{N}(R)$ is a prime ideal. Now, take $P = \langle \overline{4}XY \rangle \subseteq \mathfrak{N}(R)$. Since $P + \mathfrak{N}(R) = \mathfrak{N}(R)$ is a prime ideal, by Fact 2.1, P is a $\mathfrak{N}$-prime ideal. However, P is not a primary ideal

$$Y(\overline{4}X) = \overline{4}XY \in P,\ \overline{4}X \notin P \text{ and } Y^n \notin P \text{ for all } n \in \mathbb{N}.$$

Proposition 2.1. *For any proper ideal P of R, the followings are satisfied:*

(i) $\sqrt{P} = P + \mathfrak{N}(R)$ if P is a $\mathfrak{N}$-prime ideal.

(ii) $\sqrt{P}$ is a prime ideal if P is a $\mathfrak{N}$-prime ideal.

Proof. (i) : $P + \mathfrak{N}(R) \subseteq \sqrt{P}$ always holds. To show $\sqrt{P} \subseteq P + \mathfrak{N}(R)$, take $a \in \sqrt{P}$, then $a^n = a.a...a \in P$ for some $n \in \mathbb{N}$. Since P is a $\mathfrak{N}$-prime ideal, we obtain $a \in P + \mathfrak{N}(R)$.

(ii) : Assume that P is a $\mathfrak{N}$-prime ideal of R and $ab \in \sqrt{P}$. So $a^n b^n \in P$ for some $n \in \mathbb{N}$, then $a^n \in P + \mathfrak{N}(R) = \sqrt{P}$ or $b^n \in \sqrt{P}$ by (i). Hence, $a \in \sqrt{P}$ or $b \in \sqrt{P}$. □

It follows that every $\mathfrak{N}$-prime ideal is also a quasi primary ideal. However, a quasi primary ideal is not necessarily a $\mathfrak{N}$-prime ideal.

Example 2.3. *Consider the subring*

$$R = \{a_0 + a_1X + ... + a_nX^n : a_1 \text{ is a multiple of } 3\} \subseteq \mathbb{Z}[X]$$

and the ideal $Q = \langle 9X^2, 3X^3, X^4, X^5, X^6 \rangle$ of R. Note that $\sqrt{Q} = \langle 3X, X^2, X^3 \rangle$ and $R/\sqrt{Q} \cong \mathbb{Z}$ is an integral domain. Then Q is a quasi primary ideal, $\mathfrak{N}(R) = 0$ and $Q + \mathfrak{N}(R) = Q$. Since $9X^2 \in Q$ but $9 \notin Q + \mathfrak{N}(R)$ and $X^2 \notin Q + \mathfrak{N}(R)$, therefore, Q is not a $\mathfrak{N}$-prime ideal of R.

The following figure states the relations between $\mathfrak{N}$-prime ideals and other classical ideals

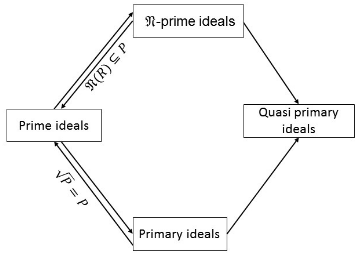

FIGURE 1. $\mathfrak{N}$-prime ideal

Corollary 2.2. *For any proper ideal P of R, the followings are equivalent:*

(i) P is a $\mathfrak{N}$-prime ideal;

(ii) $P+\mathfrak{N}(R)$ is a prime ideal of R;

(iii) $IJ \subseteq P$ implies that either

$$I \subseteq P+\mathfrak{N}(R) \text{ or } J \subseteq P+\mathfrak{N}(R)$$

for ideals I, J of R;

(iv) $(P+\mathfrak{N}(R) : a) = P+\mathfrak{N}(R)$ for every $a \notin P+\mathfrak{N}(R)$;

(v) $R/(P+\mathfrak{N}(R))$ is an integral domain.

Let R_1, R_2 be two rings (not necessarily the same), then $R = R_1 \times R_2$ becomes a commutative ring under componentwise addition and multiplication. In addition, every ideal P of R has the form $P_1 \times P_2$, where P_i is an ideal of R_i for $i = 1, 2$.

Proposition 2.2. *Let $R = R_1 \times R_2$, and $P = P_1 \times P_2$, where P_i is an ideal of R_i for $i = 1, 2$. Then the followings are equivalent:*

(i) P is a $\mathfrak{N}$-prime ideal of R.

(ii) P_1 is a $\mathfrak{N}$-prime ideal of R_1 and $P_2 = R_2$ or $P_1 = R_1$ and P_2 is a $\mathfrak{N}$-prime ideal of R_2.

Proof. $(i) \Rightarrow (ii)$: P is a $\mathfrak{N}$-prime ideal of R, by Proposition 2.1, $\sqrt{P} = \sqrt{P_1} \times \sqrt{P_2}$ is a prime ideal, so that either $P_1 = R_1$ or $P_2 = R_2$. Let $P_1 = R_1$. To prove P_2 is a $\mathfrak{N}$-prime ideal of R_2, let $ab \in P_2$, $a, b \in R_2$.

$$(0, a)(0, b) = (0, ab) \in P,$$

implies

$$(0, a) \in P+\mathfrak{N}(R) \text{ or } (0, b) \in P+\mathfrak{N}(R).$$

Hence,

$$\mathfrak{N}(R) = \mathfrak{N}(R_1) \times \mathfrak{N}(R_2)$$

and

$$P + \mathfrak{N}(R) = (P_1 + \mathfrak{N}(R_1)) \times (P_2 + \mathfrak{N}(R_2)) = R_1 \times (P_2 + \mathfrak{N}(R_2)).$$

So $a \in P_2 + \mathfrak{N}(R_2)$ or $b \in P_2 + \mathfrak{N}(R_2)$.

$(ii) \Rightarrow (i)$: Assume that $P = P_1 \times R_2$, where P_1 is a $\mathfrak{N}$-prime ideal of R_1. Then by Corollary 2.2, $R_1/(P_1 + \mathfrak{N}(R_1))$, and

$$R/(P + \mathfrak{N}(R)) \cong R_1/(P_1 + \mathfrak{N}(R_1))$$

is an integral domain. Consequently, P is a $\mathfrak{N}$-prime ideal. □

Theorem 2.1. *Let $R_1, R_2, ..., R_n$ be rings, where $n \geq 2$, and*

$$P = P_1 \times P_2 \times ... \times P_n,$$

where P_i is an ideal of R_i, $1 \leq i \leq n$. Then the followings are equivalent:

(i) P is a $\mathfrak{N}$-prime ideal of R.

(ii) P_j is a $\mathfrak{N}$-prime ideal of R_j for some $j \in \{1, 2, ..., n\}$ and $P_i = R_i$ for every $i \neq j$.

Proof. We use induction on n. By Proposition 2.2, the claim is true if $n = 2$. Assume that the claim is true for each $k \leq n-1$ and let $k = n$. Put $P' = P_1 \times P_2 \times ... \times P_{n-1}$, and $R' = R_1 \times R_2 \times ... \times R_{n-1}$, by Proposition 2.2, $P = P' \times P_n$ is a $\mathfrak{N}$-prime ideal of $R = R' \times R_n$ if and only if P' is a $\mathfrak{N}$-prime ideal of R' and $P_n = R_n$ or $P' = R'$ and P_n is a $\mathfrak{N}$-prime ideal of R_n. The rest follows from induction hypothesis. □

Corollary 2.3. *Suppose that $I \subseteq \bigcup_{i=1}^{n} P_i$ where P_i $(i = 1, ..., n)$ is a $\mathfrak{N}$-prime ideal. Then $I \subseteq P_i + \mathfrak{N}(R)$ for some $1 \leq i \leq n$.*

Proof. Since P_i $(i = 1, ..., n)$ is a $\mathfrak{N}$-prime ideal of R, by Corollary 2.2, $P_i + \mathfrak{N}(R)$ is a prime ideal for $1 \leq i \leq n$. Note that

$$I \subseteq \bigcup_{i=1}^{n} P_i \subseteq \bigcup_{i=1}^{n} (P_i + \mathfrak{N}(R)),$$

by prime avodiance lemma, we have $I \subseteq P_i + \mathfrak{N}(R)$ for some $1 \leq i \leq n$. □

Theorem 2.2. *Assume $f : R \to S$ is an epimorphism and $Ker(f) \subseteq P$ is a $\mathfrak{N}$-prime ideal. Then $f(P)$ is a $\mathfrak{N}$-prime ideal of S.*

Proof. Let P be a $\mathfrak{N}$-prime ideal of R such that $Ker(f) \subseteq P$. Let $yz \in f(P)$ for $y, z \in S$. Since f is an epimorphism, $y = f(x)$ and $z = f(t)$ for some $x, t \in R$. Then $yz = f(xt) \in f(P)$, $xt \in P$. This yields

$$x \in P + \mathfrak{N}(R) \text{ or } t \in P + \mathfrak{N}(R),$$

and so, $y \in f(P + \mathfrak{N}(R))$ or $z \in f(P + \mathfrak{N}(R))$. Since

$$f(P + \mathfrak{N}(R)) \subseteq f(P) + \mathfrak{N}(S)$$

we obtain $y \in f(P) + \mathfrak{N}(S)$ or $z \in f(P) + \mathfrak{N}(S)$. □

Corollary 2.4. *If P is a $\mathfrak{N}$-prime ideal of R that contains an ideal I, then P/I is a $\mathfrak{N}$-prime ideal of R/I.*

Proposition 2.3. *For any proper ideal P of R, the followings are satisfied:*
(i) If $\langle P, X\rangle$ is a $\mathfrak{N}$-prime ideal of $R[X]$, then P is a $\mathfrak{N}$-prime ideal of R.
(ii) If P is a $\mathfrak{N}$-prime ideal of R, then $P[X]$ is a $\mathfrak{N}$-prime ideal of $R[X]$.

Proof. (i) : Consider the homomorphism $\psi : R[X] \to R$ defined by

$$\psi(f(X)) = f(0).$$

Notice that $Ker(\psi) = \langle X\rangle \subseteq \langle P, X\rangle$ and ψ is an epimorphism. As $\langle P, X\rangle$ is a $\mathfrak{N}$-prime ideal of $R[X]$, by Theorem 2.2, $\psi(\langle P, X\rangle) = P$ is a $\mathfrak{N}$-prime ideal of R.

(ii) : Let P be a $\mathfrak{N}$-prime ideal of R. By Corollary 2.2, $R/(P + \mathfrak{N}(R))$ is an integral domain, and so is $(R/(P + \mathfrak{N}(R)))[X] \cong R[X]/(P[X] + \mathfrak{N}(R[X]))$. □

$S^{-1}R$ denotes the fractional ring of R at a multiplicatively closed subset S of R. If I is an ideal of R, then $S^{-1}I = I^e = \{\frac{a}{s} : s \in S, a \in I\}$ is an ideal of $S^{-1}R$. Furthermore, for an ideal I of R, the set $\{a \in R : ra \in I \text{ for some } r \in R - I\}$ is denoted by $Z(I)$.

Proposition 2.4. *Let P be a proper ideal of R and S be a multiplicatively closed subset of R with $S \cap P = \emptyset$. Then the followings are satisfied:*
(i) If P is a $\mathfrak{N}$-prime ideal of R, then $S^{-1}P$ is a $\mathfrak{N}$-prime ideal of $S^{-1}R$.
(ii) If $S^{-1}P$ is a $\mathfrak{N}$-prime ideal of $S^{-1}R$ with $S \cap Z(P + \mathfrak{N}(R)) = \emptyset$, then P is a $\mathfrak{N}$-prime ideal of R.

Proof. (i) : Let $\frac{a}{s}\frac{b}{t} = \frac{ab}{st} \in S^{-1}P$ for $a, b \in R; s, t \in S$. Then $uab \in P$ for some $u \in S$. Since P is a $\mathfrak{N}$-prime ideal of R, $ua \in P + \mathfrak{N}(R)$ or $b \in P + \mathfrak{N}(R)$. Hence $\frac{a}{s} = \frac{ua}{us} \in S^{-1}(P + \mathfrak{N}(R))$ or $\frac{b}{t} \in S^{-1}(P + \mathfrak{N}(R))$. Also,

$$S^{-1}(P + \mathfrak{N}(R)) = S^{-1}P + \mathfrak{N}(S^{-1}R)$$

holds.

(ii) : Let $ab \in P$ for $a, b \in R$. Then $\frac{a}{1}\frac{b}{1} \in S^{-1}P$, and $\frac{a}{1} \in S^{-1}P + \mathfrak{N}(S^{-1}R)$ or $\frac{b}{1} \in S^{-1}P + \mathfrak{N}(S^{-1}R)$. Assume that $\frac{a}{1} \in S^{-1}P + \mathfrak{N}(S^{-1}R) = S^{-1}(P + \mathfrak{N}(R))$. Then $ua \in P + \mathfrak{N}(R)$ for some $u \in S$. Since $S \cap Z(P + \mathfrak{N}(R)) = \emptyset$, we have $a \in P + \mathfrak{N}(R)$. If $\frac{b}{1} \in S^{-1}P + \mathfrak{N}(S^{-1}R)$, $b \in P + \mathfrak{N}(R)$. Hence P is a $\mathfrak{N}$-prime ideal of R. □

Let M be a unital R-module, and $R \oplus M = \{(a, m) : a \in R, m \in M\}$. Then $R \oplus M$, idealization of an R-module M, is a commutative ring with componentwise addition and the multiplication, [8]:

$$(a, m_1)(b, m_2) = (ab, am_2 + bm_1).$$

If P is an ideal of R and N is a submodule of M, then $P \oplus N$ is an ideal of $R \oplus M$ if and only if $PM \subseteq N$. Then $P \oplus N$ is called a homogeneous ideal. In [1], it was shown that $\mathfrak{N}(R \oplus M) = \mathfrak{N}(R) \oplus M$, and then all prime ideals P of $R \oplus M$ are of the form $P = P_1 \oplus M$, where P_1 is a prime ideal of R.

Theorem 2.3. *Let M be an R-module. Assume that P is an ideal of R and N is a submodule of M such that $PM \subseteq N$. Then $P \oplus N$ is a $\mathfrak{N}$-prime ideal of $R \oplus M$ if and only if P is a $\mathfrak{N}$-prime ideal of R.*

Proof. Let $P \oplus N$ be a $\mathfrak{N}$-prime ideal of $R \oplus M$, and let $ab \in P$ for $a, b \in R$. Then

$$(a, 0_M)(b, 0_M) = (ab, 0_M) \in P \oplus N.$$

This implies

$$(a, 0_M) \in P \oplus N + \mathfrak{N}(R \oplus M) \text{ or } (b, 0_M) \in P \oplus N + \mathfrak{N}(R \oplus M).$$

Thus $a \in P + \mathfrak{N}(R)$ or $b \in P + \mathfrak{N}(R)$. Suppose that P is a $\mathfrak{N}$-prime ideal of R. Then by Corollary 2.2, $R/(P + \mathfrak{N}(R))$ is an integral domain, and so

$$R \oplus M/(P \oplus N + \mathfrak{N}(R \oplus M)) \cong R/(P + \mathfrak{N}(R))$$

is an integral domain. Thus $P \oplus N$ is a $\mathfrak{N}$-prime ideal of $R \oplus M$. □

Theorem 2.4. *Let R be a ring, then the followings are equivalent:*
(i) Every ideal P of R is a $\mathfrak{N}$-prime ideal;
(ii) Every element a of R is either nilpotent or unit;
(iii) R is a quasi-local ring with (nil maximal) $\mathfrak{N}(R)$;
(iv) R is a UN-ring.

Proof. $(i) \Rightarrow (ii)$: Assume that all ideal P of R is a $\mathfrak{N}$-prime ideal of R. Since $\langle 0 \rangle$ is a $\mathfrak{N}$-prime ideal of R, by Corollary 2.2, $\langle 0 \rangle + \mathfrak{N}(R) = \mathfrak{N}(R)$ is a prime ideal of R. Let a be a nonunit element of R. Then by (i), $\langle a^2 \rangle$ is a $\mathfrak{N}$-prime ideal of R. Since $\langle a \rangle . \langle a \rangle \subseteq \langle a^2 \rangle$, we get that $\langle a \rangle \subseteq \langle a^2 \rangle + \mathfrak{N}(R)$ by Corollary 2.2. So $a = a^2x + y$ for some $x \in R, y \in \mathfrak{N}(R)$. Thus we conculde that $a - a^2x = a(1 - ax) = y \in \mathfrak{N}(R)$. As $\mathfrak{N}(R)$ is a prime ideal, we have $a \in \mathfrak{N}(R)$ or $1 - ax \in \mathfrak{N}(R)$. Assume that $1 - ax$ is nilpotent, then $1 - (1 - ax) = ax$ is a unit and hence a is a unit which is a contradiction.

$(ii) \Rightarrow (iii)$: It is clear.

$(iii) \Leftrightarrow (iv)$: It follows from [5, Proposition 2].

$(iii) \Rightarrow (i)$: Assume that R is a quasi-local ring with nil maximal ideal. Let P be a proper ideal of R. Then by assumption $P \subseteq \mathfrak{N}(R)$, and so $P + \mathfrak{N}(R) = \mathfrak{N}(R)$ is a prime ideal. By Corollary 2.2, P is a $\mathfrak{N}$-prime ideal of R. □

3. $\mathfrak{N}$-prime Spectrum of a Commutative Ring

In this section, our aim is to construct a topology on the set of all $\mathfrak{N}$-prime ideals of a ring R. We denote this set by $\mathfrak{Nspec}(\mathfrak{R})$. We examine the relations between topological properties of $\mathfrak{Nspec}(\mathfrak{R})$ and algebraic properties of R. First we define a variety of a subset $E \subseteq R$ by

$$V^*(E) := \{P \in \mathfrak{Nspec}(\mathfrak{R}) : E \subseteq \sqrt{P}\}.$$

Proposition 3.1. *Let R be a ring and $E \subseteq R$. Then the followings are satisfied:*

(i) If I is an ideal generated by the set $E \subseteq R$, then $V^*(E) = V^*(I) = V^*(\sqrt{I})$.

(ii) $V^*(0) = \mathfrak{Nspec}(\mathfrak{R})$, $V^*(R) = \emptyset$.

(iii) For each family of subsets $\{E_i\}_{i\in\Delta}$ of R, $V^*(\bigcup_{i\in\Delta} E_i) = \bigcap_{i\in\Delta} V^*(E_i)$.

(iv) For each ideals I, J of R, $V^*(I) \cup V^*(J) = V^*(I \cap J) = V^*(IJ)$.

Proof. (i) and (ii) clear.

(iii) :

$$\begin{aligned}\bigcap_{i\in\Delta} V^*(E_i) &= \{P \in \mathfrak{Nspec}(\mathfrak{R}) : E_i \subseteq \sqrt{P} \text{ for every } i \in \Delta\}\\ &= \{P \in \mathfrak{Nspec}(\mathfrak{R}) : \bigcup_{i\in\Delta} E_i \subseteq \sqrt{P}\}\\ &= V^*(\bigcup_{i\in\Delta} E_i).\end{aligned}$$

(iv) : Since $IJ \subseteq I \cap J \subseteq I, J$, $V^*(I) \cup V^*(J) \subseteq V^*(I \cap J) \subseteq V^*(IJ)$.

For the converse, take $P \in V^*(IJ)$. Then $IJ \subseteq \sqrt{P}$. Morover $\sqrt{P}$ is a prime ideal, and thus either $I \subseteq \sqrt{P}$ or $J \subseteq \sqrt{P}$. Hence $P \in V^*(I) \cup V^*(J)$. □

As a consequence of Proposition 3.1, if we assign open sets $O^*(E) = \mathfrak{Nspec}(\mathfrak{R}) - V^*(E)$, then the family $\{O^*(E) : E \subseteq R\}$ satisfies all conditions of being a topology on $\mathfrak{Nspec}(\mathfrak{R})$. We define this topology as $\mathfrak{N}$-prime spectrum of R, and denote it by $(\sigma, \mathfrak{Nspec}(\mathfrak{R}))$ or briefly $\mathfrak{Nspec}(\mathfrak{R})$. We know that zariski topology of a ring R is always a T_0-space. However $\mathfrak{Nspec}(\mathfrak{R})$ is not necessarily to be a T_0-space.

Example 3.1. *Consider the ring $\mathbb{Z}_{p^n}$ of integers modulo p^n, where p is a prime number. it is a quasi-local ring with maximal ideal $\mathfrak{N}(\mathbb{Z}_{p^n}) = \langle p\rangle$. By Theorem 2.4, every proper ideal $P = \langle p^k\rangle$ is a $\mathfrak{N}$-prime ideal of $\mathbb{Z}_{p^n}$, where $1 \le k \le n$. Moreover*

$$Spec(\mathbb{Z}_{p^n}) = \{\langle p\rangle\},\ \mathfrak{Nspec}(\mathbb{Z}_{p^n}) = \{\langle p^t\rangle : 1 \le t \le n\}.$$

Then, for any ideal $P = \langle p^k\rangle$ of $\mathbb{Z}_{p^n}$, variety of P on prime spectrum and $\mathfrak{N}$-prime spectrum are obtained $V(P) = Spec(\mathbb{Z}_{p^n})$ and $V^*(P) = \mathfrak{Nspec}(\mathbb{Z}_{p^n})$ respectively. Thus all closed subset of $\mathfrak{N}$-prime spectrum of $\mathbb{Z}_{p^n}$ is either empty or $\mathfrak{Nspec}(\mathbb{Z}_{p^n})$. Now, take singletons $\{\langle p^k\rangle\} \ne \{\langle p^t\rangle\}$, where $1 \le t \ne k \le n$. Note that all closed subset of $\mathfrak{Nspec}(\mathbb{Z}_{p^n})$ containing $\{\langle p^k\rangle\}$ also contains $\{\langle p^t\rangle\}$. Hence $\mathfrak{Nspec}(\mathbb{Z}_{p^n})$ is not a T_0-space.

Proposition 3.2. *Let R be a ring, and $X_r = X - V^*(r)$, where $X = \mathfrak{Nspec}(\mathfrak{R})$. Then $\{X_r : r \in R\}$ forms a base for $\mathfrak{N}$-prime spectrum of R.*

Proof. Let O be an open set. Then we have $O = X - V^*(E)$ for some $E \subseteq R$. Then we have

$$\begin{aligned}O &= X - V^*(E) = X - V^*(\bigcup_{r\in E}\{r\})\\ &= X - \bigcap_{r\in E} V^*(r) = \bigcup_{r\in E}(X - V^*(r)) = \bigcup_{r\in E} X_r.\end{aligned}$$

□

Proposition 3.3. *Let R be a ring, and $X_r = X - V^*(r)$, where $X = \mathfrak{Nspec}(\mathfrak{R})$.*

(i) For any $r, s \in R$, $X_{rs} = X_r \cap X_s$.

(ii) $X_r = \emptyset$ iff r is a nilpotent in R.

(iii) $X_r = X$ iff r is a unit in R.

(iv) $X_r = X_s$ iff $\sqrt{\langle r\rangle} = \sqrt{\langle s\rangle}$.

(v) X_r is quasi-compact.

(vi) X is quasi compact.

Proof. (i) : Let $P \in X_r \cap X_s$ for $P \in \mathfrak{Nspec}(\mathfrak{R})$. Then $r \notin \sqrt{P}$ and $s \notin \sqrt{P}$. Since $\sqrt{P}$ is a prime ideal, we get $rs \notin \sqrt{P}$, that is, $P \in X_{rs}$.

Conversely; let $P \in X_{rs}$. Then $rs \notin \sqrt{P}$ implies $r \notin \sqrt{P}$, and $s \notin \sqrt{P}$. This yields $P \in X_r \cap X_s$.

(ii) : Suppose that $X_r = \emptyset$, that is, $V^*(r) = \mathfrak{Nspec}(\mathfrak{R})$. Since every prime ideal is a $\mathfrak{N}$-prime, $r \in \bigcap_{P \in Spec(R)} P = \mathfrak{N}(R)$, r is a nilpotent in R. Conversely, let $r \in \mathfrak{N}(R)$ and $P \in \mathfrak{Nspec}(\mathfrak{R})$. Then by Proposition 2.1,

$$r \in \mathfrak{N}(R) \subseteq P + \mathfrak{N}(R) = \sqrt{P}$$

for any $P \in \mathfrak{Nspec}(\mathfrak{R})$, hence $P \in V^*(r)$. Therefore, $V^*(r) = \mathfrak{Nspec}(\mathfrak{R})$.

(iii) : Suppose that $X_r = X$, that is, $V^*(r) = \emptyset$. Since every maximal ideal is also a $\mathfrak{N}$-prime ideal, r is not in any maximal ideal, so that r is unit. The converse is clear.

(iv) : Suppose that $X_r = X_s$, that is, $V^*(r) = V^*(s)$. As every prime ideal of R is a $\mathfrak{N}$-prime ideal and $V^*(r) = V^*(s)$, for any $P \in Spec(R)$,

$$\langle r\rangle \subseteq P \Leftrightarrow \langle s\rangle \subseteq P.$$

So $\sqrt{\langle r\rangle} = \sqrt{\langle s\rangle}$.

Conversely, let $\sqrt{\langle r\rangle} = \sqrt{\langle s\rangle}$. Assume that $P \in V^*(r)$. Then we have $\langle r\rangle \subseteq \sqrt{P}$, and so $\langle s\rangle \subseteq \sqrt{\langle s\rangle} = \sqrt{\langle r\rangle} \subseteq \sqrt{P}$. Therefore $P \in V^*(s)$, so that $V^*(r) \subseteq V^*(s)$. Similarly $V^*(s) \subseteq V^*(r)$.

(v) : Suppose that $X_r \subseteq \bigcup_{i\in\Delta} O_i$ is an open covering. Since $\{X_r : r \in R\}$ forms a base for $\mathfrak{Nspec}(\mathfrak{R})$, we may assume that $O_i = X_{r_i}$. Then $X_r \subseteq \bigcup_{i\in\Delta} X_{r_i}$, and so $X - V^*(r) \subseteq \bigcup_{i\in\Delta} (X - V^*(r_i)) = X - \bigcap_{i\in\Delta} V^*(r_i) = X - V^*(\bigcup_{i\in\Delta}\{r_i\})$. Hence

$$V^*(\bigcup_{i\in\Delta}\{r_i\}) \subseteq V^*(r),$$

and

$$\sqrt{\langle r\rangle} \subseteq \sqrt{\langle \bigcup_{i\in\Delta}\{r_i\}\rangle}.$$

Then we have $r^n \in \langle \bigcup_{i\in\Delta} \{r_i\}\rangle$ for some $n \in \mathbb{N}$, and so $r^n = a_1r_1 + ... + a_nr_n$ for some $a_1, a_2, ..., a_n \in R$. It follows that $r^n \in \langle \bigcup_{i=1}^{n} \{r_i\}\rangle$ which implies

$$V^*(\bigcup_{i=1}^{n}\{r_i\}) \subseteq V^*(r^n) = V^*(r),$$

so that

$$X_r \subseteq X - V^*(\bigcup_{i=1}^{n}\{r_i\}) = \bigcup_{i=1}^{n} X_{r_i}.$$

(vi) : Take $r = 1$, and apply to (v). □

Note that a topological space X is called irreducible if it can not be expressed as $X = F_1 \cup F_2$, where F_1, F_2 are nonempty proper closed subsets of X.

Proposition 3.4. *Let R be a ring. The followings are equivalent:*
(i) $\mathfrak{Nspec}(\mathfrak{R})$ is an irreducible topological space.
(ii) $R/\mathfrak{N}(R)$ is an integral domain.

Proof. $(i) \Rightarrow (ii)$: Assume that $\mathfrak{Nspec}(\mathfrak{R})$ is an irreducible topological space. Let $IJ \subseteq \mathfrak{N}(R)$ for ideals I, J of R. It is clear that

$$V^*(IJ) = V^*(I) \cup V^*(J) = V^*(\mathfrak{N}(R)) = \mathfrak{Nspec}(\mathfrak{R}).$$

Then by (i), $V^*(I) = \mathfrak{Nspec}(\mathfrak{R})$ or $V^*(J) = \mathfrak{Nspec}(\mathfrak{R})$. This implies $I \subseteq \mathfrak{N}(R)$ or $J \subseteq \mathfrak{N}(R)$, that is, $\mathfrak{N}(R)$ is a prime ideal of R.

$(ii) \Rightarrow (i)$: Since $R/\mathfrak{N}(R)$ is an integral domain, $\mathfrak{N}(R)$ is a prime ideal of R. Suppose that $V^*(I) \cup V^*(J) = \mathfrak{Nspec}(\mathfrak{R})$. Then $V^*(IJ) = \mathfrak{Nspec}(\mathfrak{R})$, and $IJ \subseteq \mathfrak{N}(R)$. Therfore $I \subseteq \mathfrak{N}(R)$ or $J \subseteq \mathfrak{N}(R)$, that is, $V^*(I) = \mathfrak{Nspec}(\mathfrak{R})$ or $V^*(J) = \mathfrak{Nspec}(\mathfrak{R})$. Consequently, $\mathfrak{Nspec}(\mathfrak{R})$ is an irreducible space. □

Lemma 3.1. *Let R be a ring, and I, J be ideals of R.*
(i) $V^*(I) = V^*(J)$ if and only if $\sqrt{I} = \sqrt{J}$ for ideals I, J of R.
(ii) If $P \in \mathfrak{Nspec}(\mathfrak{R})$, then $V^*(P) = Cl(P)$.

Proof. (i) : It is clear.

(ii) : Note that $P \in V^*(P)$. Take any closed set $V^*(J)$ containing P, then $J \subseteq \sqrt{P}$. For every ideal $Q \in V^*(P)$, $P \subseteq \sqrt{Q}$, and so $J \subseteq \sqrt{P} \subseteq \sqrt{Q}$, Therfore $V^*(P)$ is the smallest closed subset of $\mathfrak{Nspec}(\mathfrak{R})$ that contains P. □

By example 3.1, $\mathfrak{N}$-prime spectrum of a ring R is not necessarily be a T_0-space. The following theorem gives the necessary and sufficient condition for $\mathfrak{N}$-prime spectrum to be a T_0-space.

Theorem 3.1. *Let R be a ring. Then every $\mathfrak{N}$-prime ideal is also a prime ideal of R if and only if $\mathfrak{Nspec}(\mathfrak{R})$ is a T_0-space.*

Proof. Suppose that every $\mathfrak{N}$-prime ideal of R is also a prime ideal. To prove that $\mathfrak{Nspec}(\mathfrak{R})$ is a T_0-space, let $Cl(P) = Cl(Q)$ for some $P, Q \in \mathfrak{Nspec}(\mathfrak{R})$. By Lemma

3.1, $V^*(Q) = V^*(P)$ and so $\sqrt{Q} = \sqrt{P}$. Then by the hypothesis, $P = Q$. Conversely, let $\mathfrak{Nspec}(\mathfrak{R})$ is a T_0-space, and $P \in \mathfrak{Nspec}(\mathfrak{R})$. Then, clearly

$$Cl(P) = V^*(P) = V^*(\sqrt{P}) = Cl(\sqrt{P}).$$

Thus $P = \sqrt{P}$ is a prime ideal by the hypothesis and Proposition 2.1. □

Theorem 3.2. *Let R be a ring. The followings are equivalent:*
(i) Every $\mathfrak{N}$-prime ideal of R is maximal;
(ii) $\mathfrak{Nspec}(\mathfrak{R})$ is a T_2-space;
(iii) $\mathfrak{Nspec}(\mathfrak{R})$ is a T_1-space.

Proof. $(i) \Rightarrow (ii)$: Suppose that every $\mathfrak{N}$-prime ideal of R is maximal, then it is prime. Hence $Spec(R)$ and $\mathfrak{Nspec}(\mathfrak{R})$ coincide. Since every prime ideal is maximal, $\mathfrak{Nspec}(\mathfrak{R}) \cong Spec(R)$ is a T_2-space.

$(ii) \Rightarrow (iii)$: It is clear.

$(iii) \Rightarrow (i)$: Assume that $\mathfrak{Nspec}(\mathfrak{R})$ is a T_1-space and $P \in \mathfrak{Nspec}(\mathfrak{R})$. By Lemma 3.1 and the hypothesis,

$$Cl(P) = V^*(P) = \{P\} = \{\sqrt{P}\} = V^*(\sqrt{P}).$$

This implies P is a maximal ideal. □

A topological space X is a connected space if it can not be express as a union of two nonempty proper disjoint closed subset of X.

Theorem 3.3. *The followings are equivalent for any ring R :*
(i) R has no proper idempotent, that is, the idempotents are 0 *and* 1.
(ii) $\mathfrak{Nspec}(\mathfrak{R})$ is a connected space.

Proof. $(i) \Rightarrow (ii)$: Assume that the only idempotents in R are 0 and 1. Suppose that $V^*(I) \cup V^*(J) = \mathfrak{Nspec}(\mathfrak{R})$ and $V^*(I) \cap V^*(J) = \emptyset$ for ideals I, J of R. Then $I + J = R$ and $IJ \subseteq \mathfrak{N}(R)$ which implies $a + b = 1$ and $(ab)^k = 0$ for some $a \in I, b \in J$ and $k \in \mathbb{N}$.
Note that $\langle a\rangle^k + \langle b\rangle^k = R$ and $\langle a\rangle^k\langle b\rangle^k = 0$, by Chinese Remainder Theorem, we get $R \cong R/\langle a\rangle^k \times R/\langle b\rangle^k$. Since R has no proper idempotent, either $R/\langle a\rangle^k = 0$ or $R/\langle b\rangle^k = 0$, that is, a is a unit or b is a unit. Hence $V^*(I) = \emptyset$ or $V^*(J) = \emptyset$. Consequently, $\mathfrak{Nspec}(\mathfrak{R})$ is a connected space.

$(ii) \Rightarrow (i)$: Suppose that $\mathfrak{Nspec}(\mathfrak{R})$ is a connected space and e is an idempotent of R. Then $e(1-e) = 0 \in \mathfrak{N}(R)$, and

$$V^*(\langle e\rangle) \cup V^*(\langle 1-e\rangle) = \mathfrak{Nspec}(\mathfrak{R}) \text{ and } V^*(\langle e\rangle) \cap V^*(\langle 1-e\rangle) = \emptyset.$$

Since $\mathfrak{Nspec}(\mathfrak{R})$ is a connected space, either $V^*(\langle e\rangle) = \mathfrak{Nspec}(\mathfrak{R})$ or $V^*(\langle e\rangle) = \emptyset$. This implies either e is a nilpotent element or a unit element, that is, $e = 0$ or $e = 1$. □

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