A NUMERICAL SOLUTION OF ANTI-PERIODIC BOUNDARY VALUE PROBLEMS VIA LEGENDRE POLYNOMIALS

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This paper introduces a numerical method to solve anti-periodic boundary value problems. The proposed method converts anti-periodic boundary value problem to a Fredholm integral equation and solves it by Galerkin method. The approximate solution converges to the exact solution and satisfies anti-periodic boundary conditions completely. Also, the convergence rate of the proposed method is given by Legendre polynomials properties.

Keywords: Anti-periodic BVP, Fredholm integral Equation, Legendre polynomials, Galerkin method.

MSC2010: 65L10, 65L60

1. Introduction

Consider the following second order differential equation

$$\begin{cases} g(t, y(t), y'(t), y''(t)) = 0, & t \in [0, T], \\ y(0) + y(T) = 0, \\ y'(0) + y'(T) = 0, \end{cases} \tag{1}$$

which is called anti-periodic boundary value problem (BVP) and the above boundary conditions are called anti-periodic boundary conditions (BC). Anti-periodic BVP has been applied by many researchers in science and engineering such as optimal control, physics and neural networks, [8, 11, 14, 18]. A number of papers address the existence and the uniqueness of solution of problem (1). For example, for a class of nonlinear second-order equations with delays is discussed in [12]. Also, many recent papers proved the existence and the uniqueness of solution of the fractional order differential version (1). The existence and uniqueness of solution of fractional order anti-periodic BVP with Laplacian operator have been presented in [7, 10]. Lv and Zhang introduced a generalized anti-periodic BVPs for fractional differential equation with p-Laplacian operator in [13]. The existence and stability of solution for systems of fractional differential equation with anti-periodic BC have been discussed in [15, 19]. A new class of anti-periodic fractional BVPs has been introduced in [3] by Ahmad et al.

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We focus on the special case (1) by the following form:

$$\begin{cases} y''(t) + p(t)y(t) = f(t), & t \in [a,b], \\ y(a) + y(b) = 0, \\ y'(a) + y'(b) = 0. \end{cases} \tag{2}$$

Aftabizadeh et al. [2] proved the existence and the uniqueness of solution of (2), when $[a,b] = [0,\pi]$, $f \in L^2[0,\pi]$ and $p \in C[0,\pi]$ with the following condition:

$$(2n-1-\delta)^2 \leq p(t) \leq (2n+1+\delta)^2, \quad 0 \leq t \leq \pi,$$

for some positive integer n and some $\delta \in (0,1)$, or

$$0 \leq p(t) \leq (1-\delta)^2, \quad 0 \leq t \leq \pi,$$

or

$$p(t) \leq 0, \quad 0 \leq t \leq \pi.$$

Note that, the above conditions can be generalized to any interval $[a,b]$. Aftabizadeh et al. [1] proved the existence and the uniqueness of solution for higher order anti-periodic BVPs. Wang and Li [16] proved only the existence of solution of (1) when

$$g(t,y,y',y'') = y'' - u(t,y),$$

u is continuous and there exist constants $0 \leq C \leq 8$ and M such that

$$|u(t,s)| \leq \frac{C}{T^2}|s| + M,$$

for all $t \in [0,T]$, $s \in \mathbb{R}$.

Aftabizadeh et al. [2] used a repeating shooting method for solving anti-periodic BVP (2). To find the suitable initial values $y(a)$ and $y'(a)$ for shooting method is very difficult. Ahmadinia and Loghmani [4] presented a numerical method for anti-periodic BVPs based on the least square method and splines. The convergence rate of the least square method by splines is constant. The present paper converts anti-periodic BVP (2) to a Fredholm integral equation of the second kind. The proposed method solves this integral equation by legendre polynomials. The properties of legendre polynomials help us to obtain the convergence rate of the proposed method. This paper is organized as follows: Section 2 describes the method for converting anti-periodic BVP (2) to Fredholm integral equation of the second kind and to solve the integral equation by Galerkin method. Section 3 proves the convergence analysis of the method and obtains the convergence rate of the approximation solution. In order to achieve this, we have applied some results of Wang and Xiang [17] on the convergence rate of Legendre polynomials, and Atkinson and Han [5] as well. The last section presents some numerical examples to confirm the theory of the presented method.

2. **Description of the Method**

Consider the following integral equation

$$u(t) = f(t) + \int_a^b K(t,s)u(s)ds, \tag{3}$$

with the kernel

$$K(t,s) := \begin{cases} W(t,s), & a < s < t, \\ G(t,s), & t < s < b, \end{cases}$$

where

$$\begin{cases} G(t,s) := \dfrac{1}{4}p(t)(b-a+2t-2s), \\ \\ W(t,s) := \dfrac{1}{4}p(t)(b-a-2t+2s). \end{cases}$$

Note that, the functions p and f are the functions used in (2). It is obvious that one can convert the Fredholm integral equation on $[a,b]$ to the same problem on $[-1,1]$ by changing variable, then without loss of generality: consider $a=-1$, $b=1$. We will solve (3) by Galerkin method with Legendre polynomial basis. Legendre polynomials as an orthogonal basis helps us to prove the convergence rate of our method. Let L_j denotes the Legendre polynomial of degree j on $[-1,1]$ as follows:

$$L_0(x) = 1, \qquad L_1(x) = x,$$
$$L_{k+1}(x) = \left(\frac{2k+1}{k+1}\right) L_k(x) - \left(\frac{x}{k+1}\right) L_{k-1}(x), \qquad k \geq 1.$$

Let Π_m be the polynomial space of degree m and

$$\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)dx, \tag{4}$$

is the inner product on Π_m. Note that Legendre polynomials are orthogonal with respect to the inner product. Consider

$$u_n(t) = \sum_{j=0}^{n} c_j L_j(t), \qquad -1 \leq t \leq 1,$$

as an approximate solution of (3) in finite dimensional space Π_n. Galerkin method obtains the unknown coefficients c_j, $\quad 0 \leq j \leq n$. So the integral equation (3) yields

$$\sum_{j=0}^{n} c_j L_j(t) = f(t) + \int_a^b K(t,s) \sum_{j=0}^{n} c_j L_j(s) ds.$$

By takeing inner product on both sides with respect to L_i, the above equation implies

$$\sum_{j=0}^{n} c_j(\langle L_j, L_i\rangle + \langle \mathcal{K} L_j, L_i\rangle) = \langle f, L_i\rangle, \qquad 0 \leq i \leq n.$$

This linear system yields the coefficients c_j, $\quad 0 \leq j \leq n$, then u_n has been obtained as an approximate solution of (3). The following function is the solution of (2), when u is the exact solution of (3),

$$y(t) = \frac{b-a+2t}{-4}\int_a^b u(s)ds + \frac{1}{2}\int_a^b su(s)ds + \int_a^t (t-s)u(s)ds. \tag{5}$$

This fact will be proved in the next section. Relation (5) and the approximate solution u_n yield an approximate solution y_n for (2).

3. The Convergence Analysis of the Method

The Fredholm integral equation of the second kind (3) has a unique solution when the following operator is compact

$$\mathcal{K}u(t) = \int_a^b K(t,s)u(s)ds, \qquad t \in [a,b], \tag{6}$$

and $I - \mathcal{K} : C[a,b] \xrightarrow[onto]{1-1} C[a,b]$ is one to one and onto operator such that $(I-\mathcal{K})^{-1}$ is a bounded operator (see Atkinson [6] page 13), that is Fredholm Alternative Theorem. The following Theorem 3.1 converts anti-periodic BVP (2) to the Fredholm integral equation of the second kind (3) and the unique solution of anti-periodic BVP (2) can be obtained by the unique solution of integral equation (3).

Theorem 3.1. *Let u be the unique solution of (3) then the defined function y in (5) is a solution of anti-periodic BVP (2). Moreover, if anti-periodic BVP (2) has the unique solution y, then $u = y''$ is the unique solution of integral equation (3).*

Proof. Let u be the unique solution of (3). Taking derivative twice in (5) yield

$$y'(t) = \frac{-1}{2}\int_a^b u(s)ds + \int_a^t u(s)ds, \tag{7}$$

$$y''(t) = u(t). \tag{8}$$

The relations (5) and (7) imply $y(a) + y(b) = 0$ and $y'(a) + y'(b) = 0$ respectively. Also, (5), (7) and (8) show that y is a solution of (2).

Moreover, if y is the unique solution of (2), taking integral on the both sides of (8) yields

$$y'(t) = y'(a) + \int_a^t u(s)ds. \tag{9}$$

Anti-periodic BC $y'(a) + y'(b) = 0$ and (9) imply

$$y'(a) = \frac{-1}{2}\int_a^b u(s)ds. \tag{10}$$

The relation (9) and (10) imply (7). The following relation is obtained by integrating of (7):

$$y(t) = y(a) + \frac{a-t}{2}\int_a^b u(s)ds + \int_a^t (t-s)u(s)ds. \tag{11}$$

Anti-periodic BC $y(a) + y(b) = 0$ and (11) yield

$$y(a) = \frac{a+b}{-4}\int_a^b u(s)ds + \frac{1}{2}\int_a^b su(s)ds. \tag{12}$$

The relation (11) and (12) imply (5). Anti-periodic problem (2) can be written as (3) by considering (5), (7) and (8). □

To prove the convergence analysis of the method, we use the following theorem which presents the convergence analysis of the approximate solution of (3).

Theorem 3.2. *Assume $\mathcal{K} : V \to V$ is bounded, with V a Banach space; and assume $\lambda - \mathcal{K} : V \xrightarrow[onto]{1-1} V$. Further assume*

$$\|\mathcal{K} - P_n\mathcal{K}\| \to 0 \quad as \quad n \to \infty$$

where P_n is a projection $P_n : V \to V_n$ and V_n is a finite dimensional space. Then for all sufficiently large n, say $n \geq N$, the operator $(\lambda - P_n\mathcal{K})^{-1}$ exists as a bounded operator form V to V. Moreover, it is uniformly bounded:

$$\sup_{n\geq N} \|(\lambda - P_n\mathcal{K})^{-1}\| < \infty.$$

For the solutions u_n with n sufficiently large and u of

$$(\lambda - P_n\mathcal{K})u_n = P_n f, \qquad u_n \in V$$

and

$$(\lambda - \mathcal{K})u = f$$

respectively, we have

$$u - u_n = \lambda(\lambda - P_n\mathcal{K})^{-1}(u - P_n u)$$

and the two-sided error estimate

$$\frac{|\lambda|}{\|\lambda - P_n\mathcal{K}\|}\|u - P_n u\| \leq \|u - u_n\| \leq |\lambda|\|(\lambda - P_n\mathcal{K})^{-1}\|\|u - P_n u\|. \tag{13}$$

Proof. This theorem has been presented by Atkinson and Han [5] page 479. □

Let V_n be the polynomial space Π_n and $V = C[a, b]$. Consider operator $\mathcal{K}$ in (6) and the following orthogonal projection P_n by the inner product (4):

$$P_n : V \to \Pi_n,$$
$$P_n f := \sum_{j=0}^{n} \frac{(2j+1)}{2}\langle f, L_j\rangle L_j.$$

The theorem 3.2 implies that $\|u-u_n\|$ converges to zero at exactly the same speed as $\|u - P_n u\|$ (it is obvious by considering (13)). Note that u is the exact solution and u_n is Galerkin approximate solution. The convergence rate of $\|u - u_n\|$ is obtained if we know that the convergence rate of $\|u - P_n u\|$. Wang and Xiang proved the convergence rate of $\|u - P_n u\|$ in [17], which helps us to obtain the convergence rate of the method.

Theorem 3.3. *If $u, u', \cdots, u^{(k-1)}$ are absolutely continuous on $[-1, 1]$ and $\|u^{(k)}\|_T = V_k < \infty$ for some $k > 1$, when*

$$\|u\|_T = \int_{-1}^{1} \frac{|u'|}{\sqrt{1 - x^2}}dx,$$

then for each $n > k + 1$,

$$\|u - P_n u\| \leq \frac{V_k}{(k-1)(n-\frac{1}{2})(n-\frac{3}{2})\cdots(n-\frac{2k-3}{2})}\sqrt{\frac{\pi}{2(n-1)}}.$$

If u is analytic inside and on the Bernstein ellipse ε_ϱ with foci ± 1 and major semiaxis and minor semiaxis summing to $\varrho > 1$, then for each $n \geq 0$,

$$\|u - P_n u\| \leq \frac{(2n\varrho + 3\varrho - 2n - 1)\ell(\varepsilon_\varrho)M}{\pi \varrho^{n+1}(\varrho - 1)^2(1 - \varrho^{-2})}.$$

where $M = \max_{z \in \varepsilon_\varrho} |f(z)|$ and $\ell(\varepsilon_\varrho)$ denotes the length of the circumference of ε_ϱ.

Proof. This theorem has been proved in [17] by Wang and Xiang. □

Theorems 3.2 and 3.3 imply that the approximate solution of (3) converges to the exact solution. The following corollary shows that the convergence rate of the proposed method. Note that, there is no theorem to state the convergence rate of $\|u - P_n u\|$ when u is analytic on the whole complex plane.

Corollary 3.1. *If u and u_n are the exact solution and Galerkin approximate solution of (3) respectively. The convergence rate of the proposed method has two cases as follows*

- *Case 1: If $u, u', \cdots, u^{(k-1)}$ are absolutely continuous on $[-1, 1]$ and $\|u^{(k)}\|_T < \infty$, then*

$$\|u - u_n\|_\infty = O\left(\frac{1}{n^{k-\frac{1}{2}}}\right), \qquad n > k + 1.$$

- *Case 2: If u is analytic inside and on ε_ϱ then*

$$\|u - u_n\|_\infty = O\left(\frac{n}{\varrho^n}\right), \qquad \varrho > 1, n \geq 0.$$

Remark 3.1. *If u is an analytic function on the whole complex plane then the convergence rate of $\|u_n - u\|$ is more than the convergence rate of case 2 in corollary 3.1.*

Remark 3.2. *Note that, the approximation solution y_n for anti-periodic BVP (2) is obtained by (5):*

$$y_n(t) = \frac{b - a + 2t}{-4} \int_a^b u_n(s)ds + \frac{1}{2} \int_a^b s u_n(s)ds + \int_a^t (t - s)u_n(s)ds,$$

consider $L = \max\{|a|, |b|\}$, the above equation and (5) yield

$$\begin{aligned} \|y_n - y\|_\infty \leq & \frac{b - a + 2L}{4} \int_a^b \|u_n - u\|_\infty ds + \frac{L}{2} \int_a^b \|u_n - u\|_\infty ds + 2L \int_a^t \|u_n - u\|_\infty ds \\ \leq & \left(\frac{b - a}{4} + 3L\right)(b - a)\|u_n - u\|_\infty. \end{aligned}$$

Hence the convergence rate of $\|y_n - y\|_\infty$ is at least the same as the convergence rate of $\|u_n - u\|_\infty$ in both cases of corollary 3.1 and the case of remark 3.1.

4. Numerical Examples and Conclusion

The present section illustrates some numerical examples which confirm the theory of the convergence of the proposed method. We need the following notations to show that the global error and convergence rate of the method.

$$E_N = \|y - y_N\|_\infty, \qquad L_N = \|y - P_N y\|_\infty,$$
$$R_N^G = \log_2\left(\frac{E_N}{E_{2N}}\right), \qquad R_N = \log_2\left(\frac{L_N}{L_{2N}}\right).$$

The exact solutions of the examples 4.1 and 4.2 are analytic functions on the whole complex plane. Considering remark 3.1, the convergence rate of the method is more than the convergence rate of case 2 in corollary 3.1. The values R_N^G and R_N show that the convergence rate of the method is the same as the convergence rate of L_N .

The convergence rate of the method for example 4.4 is $O(n/\varrho^n)$. The constant ϱ can be obtained by the following relation

$$\varrho \approx 2^{\frac{1+\log_2\left(\frac{E_N}{E_{2N}}\right)}{N}}. \tag{14}$$

The numerical results of example 4.3 show that the convergence rate of the method for this example is the third order. Case 1 in corollary 3.1 shows that the convergence rate of the method for this example is at least $O(n^{-\frac{5}{2}})$. All computations of the following examples have been run by Maple 15 software.

Example 4.1. *Consider the following anti-periodic BVP*

$$\begin{cases} y''(t) + (3+\sin(t))y(t) = f(t), & t \in [0,\pi], \\ y(0) + y(\pi) = 0, \\ y'(0) + y'(\pi) = 0, \end{cases}$$

where

$$f(t) = \sin(t)\left(\frac{t^2-\pi t}{2}\right) - \cos(t)(2+\sin(t)) + \frac{3}{2}t^2 - \frac{3}{2}\pi t + 1.$$

The exact solution is $y(t) = -\cos(t) + \frac{t^2}{2} - \frac{\pi t}{2}$. *Note that,* $p(t) = 3 + \sin(t)$ *satisfies the following condition:*

$$(2n-1-\delta)^2 \le p(t) \le (2n+1+\delta)^2, \quad 0 \le t \le \pi,$$

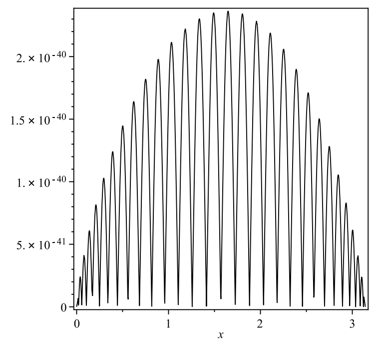
where $\delta = \frac{1}{2}$ *and* $n = 1$, *then* $y(t)$ *is the unique solution. Figures (a) and (b) show the errors for* $N = 32, 64$. *Table 1. shows the errors and convergence rates of the method for* $N = 2^j, j = 1, \ldots, 6$. *The values* R_N^G *and* R_N *are the same approximately.*

Table 1. Errors and convergence rates for example 4.1

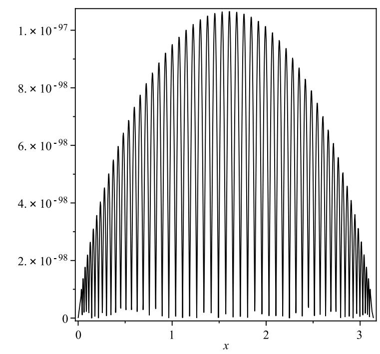
N	E_N	R_N^G	L_N	R_N
2	$1.32e-00$		$2.15e-01$	
4	$8.03e-02$	4.04	$9.03e-03$	4.57
8	$2.50e-06$	14.97	$1.55e-06$	12.47
16	$2.89e-16$	33.01	$3.28e-16$	32.16
32	$2.35e-40$	80.02	$3.92e-40$	79.44
64	$1.04e-97$	190.4	$2.60e-97$	189.96

TABLE 2. Errors and convergence rates for example 4.2

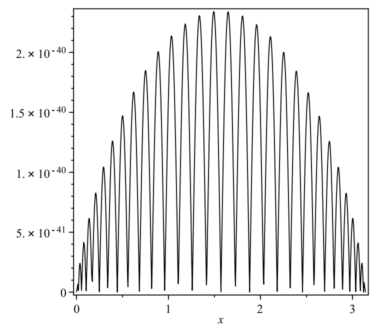
N	E_N	R_N^G	L_N	R_N
2	$9.86e-01$		$2.15e-01$	
4	$2.33e-02$	5.40	$9.03e-03$	4.57
8	$2.11e-06$	13.43	$1.58e-06$	12.47
16	$2.81e-16$	32.81	$3.28e-16$	32.16
32	$2.31e-40$	80.01	$3.98e-40$	79.44
64	$1.07e-97$	190.46	$2.60e-97$	189.96



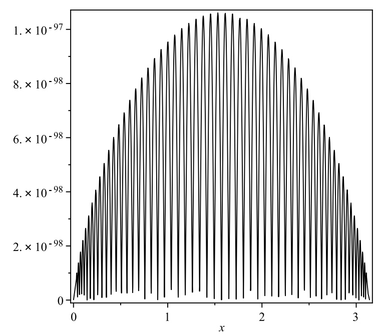
(a) $|y(t)-y_{32}(t)|$ Ex. 4.1.



(b) $|y(t)-y_{64}(t)|$ Ex. 4.1.



(c) $|y(t)-y_{32}(t)|$ Ex. 4.2.



(d) $|y(t)-y_{64}(t)|$ Ex. 4.2.

Example 4.2. *Consider the following anti-periodic BVP*

$$\begin{cases} y''(t)+\frac{\cos^2(t)}{2}y(t)=\frac{\cos^3(t)}{2}-\cos(t), & t\in[0,\pi], \\ y(0)+y(\pi)=0, & \\ y'(0)+y'(\pi)=0. & \end{cases}$$

The exact solution is $y(t)=\cos(t)$. *Note that* $p(t)=\dfrac{\cos^2(t)}{2}$ *satisfies the following condition:*

$$0\le p(t)\le(1-\delta)^2,\quad 0\le t\le\pi,$$

where $\delta=\frac{1}{2}$, *then* $y(t)$ *is the unique solution. Table 2 presents the errors and convergence order of the method and figures (c) and (d) illustrate the errors for* $N=32,64$.

Example 4.3. *Consider the following anti-periodic BVP*

$$\begin{cases} y''(t) - \exp(-t)y(t) = f(t), & t \in [-1,1], \\ y(-1) + y(1) = 0, \\ y'(-1) + y'(1) = 0. \end{cases}$$

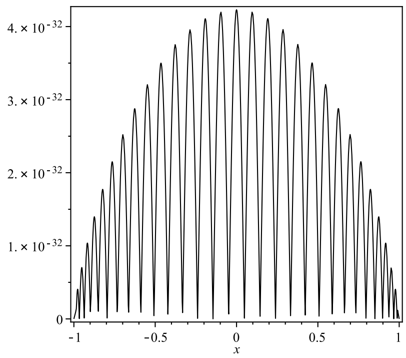
where

$$f(t) = \frac{8t^2}{(16+t^2)^3} - \frac{2}{(16+t^2)^2} - \frac{2}{17} - \exp(-t)(\frac{1}{16+t^2} - \frac{t^2}{17}).$$

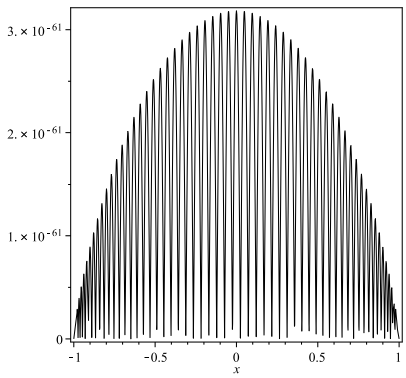
Interval $[-1,1]$ *can be changed to* $[0,\pi]$*, then* $p(t) = -exp(-(\frac{2}{\pi}t - 1)) \leq 0$ *on* $[0,\pi]$*. Then the exact solution* $y(t) = \frac{1}{16+t^2} - \frac{t^2}{17}$ *is the unique solution. Figures (e) and (f) show the errors for* $N = 32, 64$*. Table 3 presents errors and the convergence rates as well as the constant* $\varrho \approx 8.2$*. Which is obtained by (14).*

Table 3. Errors and convergence rates for example 4.3

N	E_N	R_N^G	L_N	R_N	ϱ
2	$3.07e-4$		$5.05e-05$		
4	$2.10e-06$	7.19	$9.26e-07$	5.76	10.19
8	$3.37e-10$	12.61	$2.72e-10$	11.73	9.03
16	$1.56e-17$	24.36	$1.92e-17$	23.75	8.50
32	$4.18e-32$	48.41	$7.32e-32$	47.89	8.28
64	$3.15e-61$	96.75	$7.88e-61$	96.22	8.21
128	$1.89e-119$	193.41	$6.62e-119$	193.03	8.21



(e) $|y(t) - y_{32}(t)|$ Ex. 4.3.



(f) $|y(t) - y_{64}(t)|$ Ex. 4.3 .

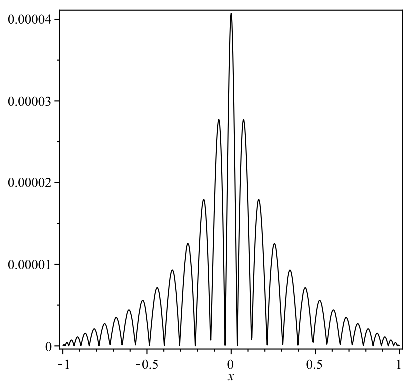
Example 4.4. *The exact solution of the following anti-periodic BVP is* $y(t) = |t|^3 - t^2$*,*

$$\begin{cases} y''(t) - (t^2+3)^{-1}y(t) = f(t), & t \in [-1,1], \\ y(-1) + y(1) = 0, \\ y'(-1) + y'(1) = 0. \end{cases}$$

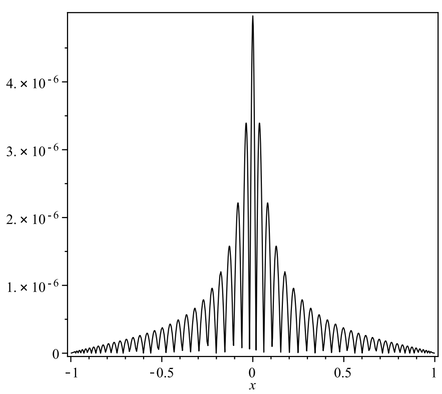
The coefficient of $y(t)$ *is* $-(t^2+3)^{-1}$, *which is negative. Then the exact solution* $y(t)$ *is unique. Figures (g) and (h) show the errors for* $N = 32, 64$. *Corollary 3.1, case 1 shows that the convergence rate of the method for Ex. 4.4 is at least* $O(n^{-\frac{5}{2}})$, *but the convergence rate of the method is the third order, as shown in Table 4.*

Table 4. Errors and convergence rates for example 4.4

N	E_N	R_N^G	L_N	R_N
2	$5.12e-1$		$1.25e-01$	
4	$3.13e-02$	4.03	$1.56e-02$	3.00
8	$3.01e-03$	3.38	$1.95e-03$	3.00
16	$3.41e-04$	3.14	$2.18e-04$	3.16
32	$4.10e-05$	3.06	$2.62e-05$	3.05
64	$4.98e-06$	3.04	$3.22e-06$	3.02

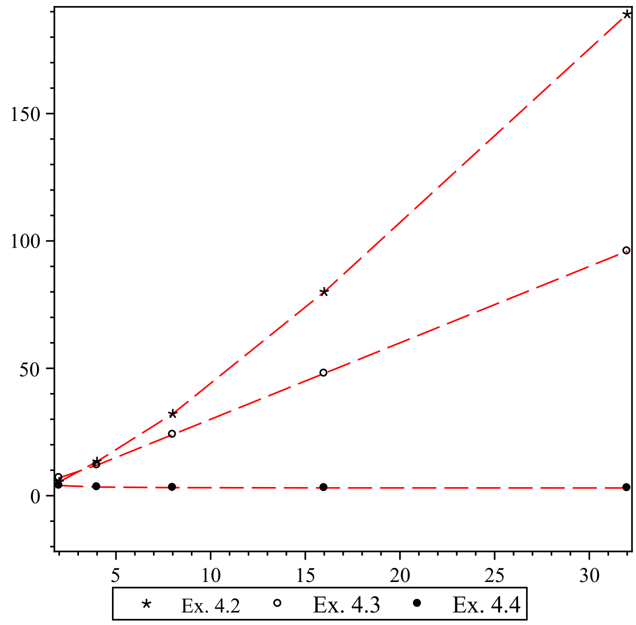


(g) $|y(t) - y_{32}(t)|$ Ex. 4.4 .



(h) $|y(t) - y_{64}(t)|$ Ex. 4.4 .

Figure (i) shows the results of tables 2, 3 and 4 which is compared the convergence rate of the method in three cases. It illustrates (N, R_N^G), $N = 2^j, j = 1, \ldots, 5$, for Ex.4.2-Ex.4.4.



(i) $(N, R_N^G), N = 2^j, j = 1, \ldots, 5$, Ex.4.2. Asterisk, Ex.4.3. Circle and Ex.4.4. Circle solid.

The convergence rate of the proposed method depends on the exact solution. If the exact solution is finitely differentiable then the order of convergence is constant. So if the exact solution is analytic inside and on the Bernstein ellipse ε_ϱ then the convergence rate of the proposed method is $O(n/\varrho^n)$. Finally if the exact solution is analytic on the whole complex plane then the convergence rate of the proposed method is more than the previous case. Aftabizadeh et al. [2] applied shooting method, but it is very difficult to find the initial values $y(a)$ and $y'(a)$. Ahmadinia and Loghmani [4] employed splines and least square method. The convergence rate of the least square method by splines is constant, and independent of the exact solution.

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