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FIXED POINT THEOREMS IN C^*-ALGEBRA-VALUED S-METRIC SPACES WITH SOME APPLICATIONS

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In this paper, we introduce the new type of metric space, namely, C^*-algebra-valued S-metric space and give some fixed point theorems for self maps with contractive conditions. As applications, existence and uniqueness results for a type of integral equation and operator equation are given.

Keywords: C^*-algebra, C^*-algebra-valued S-metric space, Fixed point, Contraction
MSC2010: 47H10, 54H25

1. Introduction

Metric spaces are very important in mathematics and applied sciences. Găhler [4] and Dhage [3] introduced the concepts of 2-metric spaces and D-metric spaces, respectively, Mustafa and Sims [10] introduced a new structure of generalized metric spaces which are called G-metric spaces as a generalization of metric spaces. Further, Sedghi, Shobe and Zhou [11] introduced the notion of a D^* -metric space as an improved version of a Dhage D-metric space. Later, he examined the shortcomings of both G-metric and D^*-metric spaces and gave the concept of a new generalized metric space called a S-metric space [12] and investigated some of their properties. Also discussed a fixed point theorem for self mapping on a complete S-metric space.

Now, we present some necessary concepts and results in C^*-algebra [2, 9]. Suppose that $\mathcal{A}$ is an unital algebra. An involution on $\mathcal{A}$ is a conjugate-linear map $a \mapsto a^*$ on $\mathcal{A}$ such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{A}$. The pair $(\mathcal{A}, *)$ is called a $*$-algebra. A Banach $*$-algebra is a $*$-algebra $\mathcal{A}$ together with a complete sub-multiplicative norm such that $\|a^*\| = \|a\| (\forall a \in \mathcal{A})$. A C^*-algebra is a Banach $*$-algebra such that $\|a^*a\| = \|a\|^2$ [9]. It is clear that under the norm topology, $L(H)$, the set of all bounded linear operators on a Hilbert space H, is a C^*-algebra. In [7], Ma established the notion of C^*- algebra-valued metric spaces and proved some fixed point theorems for self maps with contractive or expansive mappings. The main idea consists in using the set of all positive elements of a unital C^*- algebra instead of the set of real numbers. Further in [6, 8], introduced a concept of C^*- algebra-valued b-metric spaces which generalizes the concept of C^*-algebra valued metric spaces.

In this paper, we introduce the new type of metric space, namely, C^*- algebra valued S-metric space and give some fixed point theorems for self maps with contractive conditions. Some applications of our obtained results are given.

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2. Preliminaries

Throughout this paper, $\mathcal{A}$ will denote an unital C^*-algebra. Set $\mathcal{A}_h = \{a \in \mathcal{A} : a = a^*\}$. We call an element $a \in \mathcal{A}$ a positive element, denote it by $0_\mathcal{A} \preceq a$, if $a = a^*$ and $\sigma(a) \subseteq [0, \infty)$, where $0_\mathcal{A}$ is zero element in $\mathcal{A}$ and $\sigma(a)$ is the spectrum of a. There is a natural partial ordering on $\mathcal{A}_h$ given by $a \preceq b$ if and only if $0_\mathcal{A} \preceq b - a$. From now on, $\mathcal{A}_+$ and $\mathcal{A}^{'}$ will denote the set $\{a \in \mathcal{A} : 0_\mathcal{A} \preceq a\}$ and the set $\{a \in \mathcal{A} : ab = ba, \forall b \in \mathcal{A}\}$, respectively.

Lemma 2.1. [9] *Suppose that $\mathcal{A}$ is a unital C^*-algebra with a unit 1_A.*

(1) *For any $x \in \mathcal{A}_+$, we have $x \preceq 1_A \Leftrightarrow \|x\| \leq 1$.*
(2) *If $a \in \mathcal{A}_+$ with $\|a\| < \frac{1}{2}$, then $1_A - a$ is invertible and $\|a(1_A - a)^{-1}\| < 1$.*
(3) *Suppose that $a, b \in \mathcal{A}$ with $a, b \succeq 0_\mathcal{A}$ and $ab = ba$, then $ab \succeq 0_\mathcal{A}$*
(4) *Let $a \in \mathcal{A}^{'}$, if $b, c \in \mathcal{A}$ with $b \succeq c \succeq 0_\mathcal{A}$, and $1_A - a \in \mathcal{A}^{'}_+$ is an invertible operator, then $(1_A - a)^{-1} b \succeq (1_A - a)^{-1} c$.*

Definition 2.1. [7] *Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathcal{A}$ Satisfies:*

(1) *$0_\mathcal{A} \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0_\mathcal{A}$ iff $x = y$;*
(2) *$d(x, y) = d(y, x)$ for all $x, y \in X$;*
(3) *$d(x, y) \preceq d(x, z) + d(y, z)$ for all $x, y, z \in X$.*

Then d is called a C^*- algebra-valued metric on X and $(X, \mathcal{A}, d)$ is said to be a C^*-algebra-valued metric space.
Now, We introduce the notion of C^*-algebra-valued S-metric space.

Definition 2.2. *Let X be a nonempty set. Suppose the mapping $S : X \times X \times X \to \mathcal{A}$ Satisfies:*

(S1) *$0_\mathcal{A} \preceq S(x, y, z)$ for all $x, y, z \in X$;*
(S2) *$S(x, y, z) = 0_\mathcal{A}$ if and only if $x = y = z$;*
(S3) *$S(x, y, z) \preceq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.*

Then S is said to be C^*- algebra-valued S-metric on X and $(X, \mathcal{A}, S)$ is said to be a C^*-algebra-valued S -metric space.

Definition 2.3. *Suppose that $(X, \mathcal{A}, S)$ be a C^*-algebra-valued S-metric space. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. If $S(x_n, x_n, x) \overset{\|.\|_\mathcal{A}}{\to} 0_\mathcal{A}$ [i.e. $\|S(x_n, x_n, x)\| \to 0$] $(n \to \infty)$, then it is said that $\{x_n\}$ converges to x, and we denote it by $\lim_{n\to\infty}\{x_n\} = x$.*

If for any $p \in N$, $S(x_{n+p}, x_{n+p}, x_n) \overset{\|.\|_\mathcal{A}}{\to} 0_\mathcal{A}$ $(n \to \infty)$, then $\{x_n\}$ is called a Cauchy sequence in X.

If every Cauchy sequence is convergent in X, then $(X, \mathcal{A}, S)$ is called a complete C^*-algebra-valued S-metric space.

Example 2.1.

Let $X = \mathbb{R}$ and $\mathcal{A} = M_2(\mathbb{C})$, the set of bounded linear operators on a Hilbert space $\mathbb{C}^2$. Define $S : X \times X \times X \to \mathcal{A}$ by

$$S(x, y, z) = \begin{bmatrix} |x - z| + |y - z| & 0 \\ 0 & k|x - z| + |y - z| \end{bmatrix},$$

where $k > 0$ is a constant. Then $(X, \mathcal{A}, S)$ is a complete C^*-algebra-valued S-metric space.

Lemma 2.2. [13]

(1) If $\{b_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ and $\lim_{n\to\infty} b_n = 0_\mathcal{A}$, then for any $a \in \mathcal{A}$, $\lim_{n\to\infty} a^* b_n a = 0_\mathcal{A}$.

(2) If $a, b \in \mathcal{A}_h$ and $c \in \mathcal{A}'_+$, then $a \preceq b$ deduces $ca \preceq cb$, where $\mathcal{A}'_+ = \mathcal{A}_+ \cap \mathcal{A}'$.

Lemma 2.3. *Let $(X, \mathcal{A}, S)$ be a complete C^*-algebra-valued S-metric space. Then, $S(x, x, y) = S(y, y, x)$.*

Proof. By (S3), we get,

$$\begin{aligned} S(x,x,y) &\preceq S(x,x,x) + S(x,x,x) + S(y,y,x) = S(y,y,x) &(1)\\ S(y,y,x) &\preceq S(y,y,y) + S(y,y,y) + S(x,x,y) = S(x,x,y). &(2)\end{aligned}$$

From (1) and (2), $S(x, x, y) = S(y, y, x)$. □

Lemma 2.4. *Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converge to x and y, respectively, then $x = y$.*

Proof. Let $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$. By (S3) and by lemma 2.3, we have

$$\begin{aligned} S(x,x,y) &\preceq S(x,x,x_n) + S(x,x,x_n) + S(y,y,x_n)\\ &= S(x_n,x_n,x) + S(x_n,x_n,x) + S(x_n,x_n,y)\\ &= 2S(x_n,x_n,x) + S(x_n,x_n,y) \longrightarrow 0_A \ (n \to \infty).\end{aligned}$$

Therefore, $\|S(x, x, y)\| = 0 \Leftrightarrow x = y$. □

3. Main result

Theorem 3.1. *Let $(X, \mathcal{A}, S)$ be a complete C^*-algebra-valued S-metric space. Suppose that the mapping $f : X \to X$ satisfies $S(fx, fx, fy) \preceq a^* S(x, x, y)a$, where $a \in \mathcal{A}'_+$ with $\|a\| < 1$, for all $x, y \in X$. Then there exist a unique fixed point in X.*

Proof. Without loss of generality, one can suppose that $a \neq 0_{\mathcal{A}}$. Let $x_0 \in X$. Construct a sequences $\{x_n\} \subseteq X$ such that $x_{n+1} = fx_n$. By using (3),

$$\begin{aligned} S(x_{n+1},x_{n+1},x_n) &= S(fx_n,fx_n,fx_{n-1}) \preceq a^*S(x_n,x_n,x_{n-1})a\\ &\preceq (a^*)^2 S(x_{n-1},x_{n-1},x_{n-2})a^2\\ &\dots\\ &\preceq (a^*)^n S(x_1,x_1,x_0)a^n,\end{aligned}$$

where we use the property, if $b, c \in \mathcal{A}_h$, then $b \preceq c$ implies $a^*ba \preceq a^*ca$ [9].

For any $n + 1 > m$ and by Lemma 2.3,

$$\begin{aligned} S(x_{n+1},x_{n+1},x_m) &\preceq S(x_{n+1},x_{n+1},x_n) + S(x_{n+1},x_{n+1},x_n) + S(x_m,x_m,x_n)\\ &= 2S(x_{n+1},x_{n+1},x_n) + S(x_n,x_n,x_m)\\ &\preceq 2S(x_{n+1},x_{n+1},x_n) + [S(x_n,x_n,x_{n-1})\\ &\quad + S(x_n,x_n,x_{n-1}) + S(x_m,x_m,x_{n-1})]\\ &\preceq 2S(x_{n+1},x_{n+1},x_n) + 2S(x_n,x_n,x_{n-1}) +\\ &\quad \dots + 2S(x_{m+1},x_{m+1},x_m)\\ &\preceq 2\sum_{k=m}^{n}(a^*)^k S(x_1,x_1,x_0)a^k\end{aligned}$$

Let $S(x_1, x_1, x_0) = c$ for some $c \in A'_+$.

$$\begin{aligned} S(x_{n+1},x_{n+1},x_m) &\preceq 2\sum_{k=m}^{n}(a^*)^k c^{\frac12}c^{\frac12}a^k = 2\sum_{k=m}^{n}(c^{\frac12}a^k)^*(c^{\frac12}a^k)\\ &\preceq 2\sum_{k=m}^{n}|c^{\frac12}a^k|^2 \preceq 2\sum_{k=m}^{n}\|c^{\frac12}a^k\|^2 1_A\end{aligned}$$

$$\preceq \quad 2\|c^{\frac{1}{2}}\|^2 1_A \sum_{k=m}^{n} \|a\|^{2k} \preceq 2\|c^{\frac{1}{2}}\|^2 1_A \frac{\|a\|^{2m}}{1-\|a\|^2} \longrightarrow 0_{\mathcal{A}} \ (m \to \infty).$$

From Definition 2.3, we get that $\{x_n\}$ is a Cauchy sequence with respect to $\mathcal{A}$. The completion of X implies that there exist $x \in X$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} fx_{n-1} = x$.

Using the condition (S3) and by Lemma 2.3,

$$\begin{aligned} S(x,x,fx) &\preceq S(x,x,fx_n) + S(x,x,fx_n) + S(fx,fx,fx_n) \\ &= 2S(x,x,fx_n) + S(fx_n,fx_n,fx) \\ &\preceq 2S(x,x,fx_n) + a^*S(x_n,x_n,x)a \longrightarrow 0_{\mathcal{A}} \ (n \to \infty). \end{aligned}$$

Hence, $fx = x$. i.e. x is a fixed point of f.

Uniqueness: Suppose that $y(\neq)x$, is another fixed point of f. Then,

$$0_{\mathcal{A}} \quad \preceq \quad S(x,x,y) = S(fx,fx,fy) \preceq a^*S(x,x,y)a$$

which together with $\|a\| < 1$ yields that

$$\begin{aligned} 0 \le \|S(x,x,y)\| &\le \|a^*S(fx,fx,fy)a\| \le \|a^*\|\|S(x,x,y)\|\|a\| \\ &\le \|a\|^2\|S(x,x,y)\| < \|S(x,x,y)\|. \end{aligned}$$

Thus, $\|S(x,x,y)\| = 0$ and $S(x,x,y) = 0_{\mathcal{A}}$, which gives $x = y$. □

Corollary 3.1. *Let $(X, \mathcal{A}, S)$ be a complete C^*-algebra-valued S-metric space. Suppose that the mapping $f : X \to X$ satisfies*

$$\|S(fx,fx,fy)\| \le \|a\| \, \|S(x,x,y)\| \tag{3}$$

where $a \in \mathcal{A}'_+$ with $\|a\| < 1$, for all $x, y \in X$. Then there exist a unique fixed point in X.

Theorem 3.2. *Let $(X, \mathcal{A}, S)$ be a complete C^*-algebra-valued S-metric space. Suppose the mapping, $f : X \to X$ satisfies*

$$S(fx,fx,fy) \preceq a(S(fx,fx,x) + S(fy,fy,y)) \tag{4}$$

where $a \in \mathcal{A}'_+$ and $\|a\| < \frac{1}{2}$, $\forall x, y \in X$. Then there exists a unique fixed point in X.

Proof. Without loss of generality, one can suppose that $a \neq 0_A$. Let $x_0 \in X$. Construct a sequences $\{x_n\} \subseteq X$ such that $x_{n+1} = fx_n$. For any $n \in N$ and by using (4),

$$\begin{aligned} S(x_{n+1},x_{n+1},x_n) &= S(fx_n,fx_n,fx_{n-1}) \\ &\preceq a[S(fx_n,fx_n,x_n) + S(fx_{n-1},fx_{n-1},x_{n-1})] \\ &= a[S(x_{n+1},x_{n+1},x_n) + S(x_n,x_n,x_{n-1})] \\ (1_A - a)S(x_{n+1},x_{n+1},x_n) &\preceq aS(x_n,x_n,x_{n-1}) \\ S(x_{n+1},x_{n+1},x_n) &\preceq (1_A - a)^{-1}aS(x_n,x_n,x_{n-1}) \\ &= bS(x_n,x_n,x_{n-1}) \preceq b^2S(x_{n-1},x_{n-1},x_{n-2}) \\ &\cdots \\ &\preceq S(x_1,x_1,x_0). \end{aligned}$$

where $b = (1_A - a)^{-1}a$ and by using Lemma 2.1, we have $a \in \mathcal{A}'_+$ with $\|a\| < \frac{1}{2}$, one have $(1_A - a)^{-1} \in \mathcal{A}'_+$ and $a(1_A - a)^{-1} \in \mathcal{A}'_+$ with $\left\|a(1_A - a)^{-1}\right\| < 1$.

Let $S(x_1, x_1, x_0) = c$, $c \in A'_+$. For any $n+1 > m$,

$$\begin{aligned}
S(x_{n+1}, x_{n+1}, x_m) &\preceq S(x_{n+1}, x_{n+1}, x_n) + S(x_{n+1}, x_{n+1}, x_n) + S(x_m, x_m, x_n) \\
&= 2S(x_{n+1}, x_{n+1}, x_n) + S(x_n, x_n, x_m) \\
&\preceq 2S(x_{n+1}, x_{n+1}, x_n) + [S(x_n, x_n, x_{n-1}) \\
&\quad + S(x_n, x_n, x_{n-1}) + S(x_m, x_m, x_{n-1})] \\
&\preceq 2S(x_{n+1}, x_{n+1}, x_n) + 2S(x_n, x_n, x_{n-1}) \\
&\quad + \ldots + 2S(x_{m+1}, x_{m+1}, x_m) \\
&\preceq 2[b^n + b^{n-1} + \ldots + b^m] S(x_1, x_1, x_0) \\
&\preceq 2\sum_{k=m}^{n} b^k c = 2\sum_{k=m}^{n} b^{\frac{k}{2}} b^{\frac{k}{2}} c^{\frac{1}{2}} c^{\frac{1}{2}} = 2\sum_{k=m}^{n} c^{\frac{1}{2}} b^{\frac{k}{2}} b^{\frac{k}{2}} c^{\frac{1}{2}} \\
&= 2\sum_{k=m}^{n} (b^{\frac{k}{2}} c^{\frac{1}{2}})^* (b^{\frac{k}{2}} c^{\frac{1}{2}}) = 2\sum_{k=m}^{n} |b^{\frac{k}{2}} c^{\frac{1}{2}}|^2 \\
&\preceq 2\|\sum_{k=m}^{n} |b^{\frac{k}{2}} c^{\frac{1}{2}}|^2\| 1_A \preceq 2\|c^{\frac{1}{2}}\|^2 1_A \sum_{k=m}^{n} \|b^k\|^2 \preceq 2\|c^{\frac{1}{2}}\|^2 1_A \sum_{k=m}^{n} \|b\|^{2k} \\
&\preceq 2\|c^{\frac{1}{2}}\|^2 \frac{\|b\|^{2m}}{1-\|b\|^2} 1_A \longrightarrow 0_{\mathcal{A}} \ (m \to \infty.)
\end{aligned}$$

From Definition 2.3, we get that $\{x_n\}$ is a Cauchy sequence with respect to $\mathcal{A}$. The completion of X implies that there exist $x \in X$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} fx_{n+1} = x$.

Using (S3) and by Lemma 2.3,

$$\begin{aligned}
S(fx, fx, x) &\preceq S(fx, fx, fx_n) + S(fx, fx, fx_n) + S(x, x, fx_n) \\
&= 2(S(fx, fx, fx_n) + S(fx_n, fx_n, x) \\
&\preceq 2[a(S(fx, fx, x) + S(fx_n, fx_n, x_n)] + S(fx_n, fx_n, x) \\
(1_A - 2a)S(fx, fx, x) &\preceq 2aS(fx_n, fx_n, x_n) + S(fx_n, fx_n, x) \\
S(fx, fx, x) &\preceq 2a(1_A - 2a)^{-1} S(fx_n, fx_n, x_n) \\
&\quad + (1_A - 2a)^{-1} S(fx_n, fx_n, x) \\
\|S(fx, fx, x)\| &\leq \|2a(1_A - 2a)^{-1}\| \|S(fx_n, fx_n, x_n)\| \\
&\quad + \|(1_A - 2a)^{-1}\| \|S(fx_n, fx_n, x)\| \longrightarrow 0 \ (n \to \infty).
\end{aligned}$$

Therefore $fx = x$. i.e. x is a fixed point of f.
Uniqueness:
Suppose that $y(\neq)x$, is another fixed point of f. Then,

$$0_{\mathcal{A}} \preceq S(x, x, y) = S(fx, fx, fy) \preceq a[S(fx, fx, x) + s(fy, fy, y)] = 0_{\mathcal{A}}.$$

Hence, $S(x, x, y) = 0_{\mathcal{A}} \Leftrightarrow x = y$. Therefore the fixed point is unique. □

Theorem 3.3. *Let $(X, \mathcal{A}, S)$ be a complete C^*-algebra-valued S metric space. Suppose that the mapping $f : X \to X$ satisfies*

$$S(fx, fx, fy) \preceq a(S(fx, fx, y) + S(fy, fy, x)) \tag{5}$$

where $a \in \mathcal{A}'_+$ and $\|a\| < \frac{1}{2}$, $\forall x, y \in X$. Then there exists a unique fixed point in X.

Proof. Without loss of generality, one can suppose that $a \neq 0_A$. Let $x_0 \in X$. Construct a sequences $\{x_n\} \subseteq X$ such that $x_{n+1} = fx_n$. By using (6), for any $n \in N$,

$$\begin{aligned} S(x_{n+1}, x_{n+1}, x_n) &= S(fx_n, fx_n, fx_{n-1}) \\ &\preceq a[S(fx_n, fx_n, x_{n-1}) + S(fx_{n-1}, fx_{n-1}, x_n)] \\ &\preceq a[S(fx_n, fx_n, fx_{n-2}) + S(fx_{n-1}, fx_{n-1}, fx_{n-2})] \\ &\preceq a[S(fx_n, fx_n, fx_{n-1}) + S(fx_n, fx_n, fx_{n-1}) \\ &\quad +S(fx_{n-2}, fx_{n-2}, fx_{n-1}] \\ &\preceq a[2S(fx_n, x_n, fx_{n-1}) + S(fx_{n-1}, fx_{n-1}, fx_{n-2}] \\ &\preceq a[2S(x_{n+1}, x_{n+1}, x_n) + S(x_n, x_n, x_{n-1}] \\ (1_A - 2a)S(x_{n+1}, x_{n+1}, x_n) &\preceq a[S(x_n, x_n, x_{n-1}) \end{aligned}$$

By using Lemma 2.1, we have $a \in \mathcal{A}_+^{'}$ with $\|a\| < \frac{1}{2}$, one have $(1_A - a)^{-1} \in \mathcal{A}_+^{'}$ and $a(1_A - a)^{-1} \in \mathcal{A}_+^{'}$ with $\left\|a(1_A - a)^{-1}\right\| < 1$. Therefore,

$$S(x_{n+1}, x_{n+1}, x_n) \preceq (1_A - 2a)^{-1} aS(x_n, x_n, x_{n-1}) = bS(x_n, x_n, x_{n-1}),$$

where $b = (1_A - a)^{-1}a$. For $n + 1 > m$,

$$\begin{aligned} S(x_{n+1}, x_{n+1}, x_m) &\preceq S(x_{n+1}, x_{n+1}, x_n) + S(x_{n+1}, x_{n+1}, x_n) + S(x_m, x_m, x_n) \\ &= 2S(x_{n+1}, x_{n+1}, x_n) + S(x_n, x_n, x_m) \\ &\preceq 2S(x_{n+1}, x_{n+1}, x_n) + [S(x_n, x_n, x_{n-1}) \\ &\quad +S(x_n, x_n, x_{n-1}) + S(x_m, x_m, x_{n-1})] \\ &\preceq 2S(x_{n+1}, x_{n+1}, x_n) + 2S(x_n, x_n, x_{n-1}) \\ &\quad + \ldots + 2S(x_{m+1}, x_{m+1}, x_m) \\ &\preceq 2[b^n + b^{n-1} + \ldots + b^m]S(x_1, x_1, x_0) \\ &\preceq 2\sum_{k=m}^{n} b^k c = 2\sum_{k=m}^{n} b^{\frac{k}{2}} b^{\frac{k}{2}} c^{\frac{1}{2}} c^{\frac{1}{2}} \\ &= 2\sum_{k=m}^{n} c^{\frac{1}{2}} b^{\frac{k}{2}} b^{\frac{k}{2}} c^{\frac{1}{2}} = 2\sum_{k=m}^{n} (b^{\frac{k}{2}} c^{\frac{1}{2}})^* (b^{\frac{k}{2}} c^{\frac{1}{2}}) = 2\sum_{k=m}^{n} |b^{\frac{k}{2}} c^{\frac{1}{2}}|^2 \\ &\preceq 2\|\sum_{k=m}^{n} |b^{\frac{k}{2}} c^{\frac{1}{2}}|^2\| 1_A \preceq 2\|c^{\frac{1}{2}}\|^2 1_A \sum_{k=m}^{n} \|b^k\|^2 \\ &\preceq 2\|c^{\frac{1}{2}}\|^2 1_A \sum_{k=m}^{n} \|b\|^{2k} \preceq 2\|c^{\frac{1}{2}}\|^2 \frac{\|b\|^{2m}}{1 - \|b\|^2} 1_A \longrightarrow 0_A \ (m \to \infty). \end{aligned}$$

From Definition 2.3, we get that $\{x_n\}$ is a Cauchy sequence with respect to $\mathcal{A}$. The completion of X implies that there exist $x \in X$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} fx_{n-1} = x$. Using

(S3) and by Lemma 2.3,

$$\begin{aligned}
S(fx,fx,x) &\preceq S(fx,fx,fx_n)+S(fx,fx,fx_n)+S(x,x,fx_n)\\
&= 2(S(fx,fx,fx_n)+S(fx_n,fx_n,x)\\
&\preceq 2[a(S(fx,fx,x_n)+S(fx_n,fx_n,x)]\\
&\quad +S(x_{n+1},x_{n+1},x)\\
&= 2a(S(fx,fx,x_n)+2aS(fx_n,fx_n,x)\\
&\quad +S(x_{n+1},x_{n+1},x)\\
&\preceq 2a[S(fx,fx,x)+S(fx,fx,x)+S(x_n,x_n,x)]\\
&\quad +2aS(x_{n+1},x_{n+1},x)+S(x_{n+1},x_{n+1},x)\\
&= 4aS(fx,fx,x)+2aS(x_n,x_n,x)\\
&\quad +2aS(x_{n+1},x_{n+1},x)+S(x_{n+1},x_{n+1},x)\\
(1_A-4a)S(fx,fx,x) &\preceq 2aS(x_n,x_n,x)+2aS(x_{n+1},x_{n+1},x)\\
&\quad +S(x_{n+1},x_{n+1},x)\\
\|S(fx,fx,x)\| &\leq \|2a(1_A-4a)^{-1}\|[\|S(x_n,x_n,x)\|+\|S(x_{n+1},x_{n+1},x)\|]\\
&\quad +\|(1_A-4a)^{-1}\|\|S(x_{n+1},x_{n+1},x)\| \longrightarrow 0_{\mathcal{A}}\ (n\to\infty).
\end{aligned}$$

Therefore $fx=x$. i.e) x is a fixed point of f.
Uniqueness: Suppose that $y(\neq)x$, is another fixed point of f.

$$0_{\mathcal{A}} \preceq S(x,x,y)=S(fx,fx,fy)\preceq a[S(fx,fx,y)+s(fy,fy,x)]\preceq \quad a[S(x,x,y)+s(y,y,x)]$$

$$\begin{aligned}
(1_A-a)S(x,x,y) &\preceq aS(y,y,x)\\
S(x,x,y) &\preceq a(1_A-a)^{-1}S(y,y,x)\preceq a(1_A-a)^{-1}S(x,x,y).
\end{aligned}$$

Since $\|a(1_A-a)^{-1}\|<1$,

$$\begin{aligned}
0 &\leq \|S(x,x,y)\|=\|S(fx,fx,fy)\|\leq\|a(1_A-a)^{-1}S(x,x,y)\|\\
&\leq \|a(1_A-a)^{-1}\|\|S(x,x,y)\|<\|S(x,x,y)\|
\end{aligned}$$

This means that, $S(x,x,y)=0_{\mathcal{A}}\Leftrightarrow x=y$. Therefore the fixed point is unique. □

Example 3.1.

Let $X=L^\infty(E)$ and $H=L^2(E)$, where E is Lebesque measurable set. By $L(H)$ we denote the set of bounded linear operators on Hilbert space H. Clearly $L(H)$ is a C^*-algebra with usual operator norm.

Define $S: X\times X\times X\to L(H)$ by $S(f,g,p)=\pi_{|f-p|+|g-p|}$ $(\forall f,g,p\in X)$, where $\pi_h: H\to H$ is multiplication operator, $\pi_h(\phi)=h\cdot\phi$, for $\phi\in H$. Then S is a C^*-algebra-valued S-metric and $(X,L(H),S)$ is a complete C^*-algebra-valued S-metric space.

Let $\{f_n\}_{n=1}^\infty$ in X be a Cauchy sequence with respect to $L(H)$. i.e) for any $p\in N$, $\|S(f_{n+p},f_{n+p},f_n)\|\to 0$ $(n\to\infty)$. Now,

$$\begin{aligned}
\|S(f_{n+p},f_{n+p},f_n)\| &= \left\|\pi_{|f_{n+p}-f_n|+|f_{n+p}-f_n|}\right\|\\
&= \left\|\pi_{2|f_{n+p}-f_n|}\right\|=\|2(f_{n+p}-f_n)\|_\infty\to 0\ (n\to\infty).
\end{aligned}$$

Then $\{f_n\}_{n=1}^\infty$ is Cauchy sequence in the space X. Since X is complete C^*-algebra-valued S-metric space, there exists $f\in X$ such that $\|f_n-f\|_\infty\to 0$ $(n\to\infty)$. Therefore,

$$\begin{aligned}
\|S(f_n,f_n,f)\| &= \left\|\pi_{|f_n-f|+|f_n-f|}\right\|=\|2(f_n-f)\|_\infty\\
&= \|(f_n-f)\|_\infty+\|(f_n-f)\|_\infty\to 0\ (n\to\infty).
\end{aligned}$$

Hence the sequence $\{f_n\}_{n=1}^\infty$ converges to the function f in X with respect to $L(H)$. Thus, $(X,L(H),S)$ is complete with respect to $L(H)$.

4. **Applications**

As applications of contractive mapping theorem on complete C^*-algebra valued S-metric space, existence and uniqueness results for a type of integral equation and operator equation are given.

Example 4.1. *Consider the integral equation*

$$x(t) = \int_E K(t, s, x(s))ds + g(t), t \in E,$$

where E is the Lebesgue measurable set. Assume that the following hypotheses hold

(1) $K : E \times E \times \mathbb{R} \to \mathbb{R}$ *integrable and* $g \in L^\infty(E)$*;*
(2) *there exists a continuous function* $\phi : E \times E \to \mathbb{R}$ *and* $k \in (0, 1)$ *such that*

$$|K(t, s, u) - K(t, s, v)| \leq k\, |\phi(t, s)(u - v)|\, ,$$

for $t, s \in E$ *and* $u, v \in \mathbb{R}$*;*
(3) $sup_{t\in E} \int_E |\phi(t, s)|\, ds \leq 1$.

Then the integral equation has a unique solution x^* *in* $L^\infty(E)$.

Proof. Let $X = L^\infty(E)$ and $H = L^2(E)$, where E is Lebesque measurable set. Set S as Example 3.1, then S is a C^*-algebra valued S-metric and $(X, L(H), S)$ is a complete C^*-algebra valued S-metric space with respect to $L(H)$.

Let $T : L^\infty(E) \to L^\infty(E)$ be

$$Tx(t) = \int_E K(t, s, x(s))ds + g(t), t \in E.$$

Set $A = kI$, then $A \in L(H)_+$ and $\|A\| = k < 1$. For any $h \in H$,

$$\begin{aligned}
\|S(Tx, Tx, Ty)\| &= sup_{\|h\|=1} \left\langle \pi_{(|Tx-Ty|+|Tx-Ty|)} h, h \right\rangle \\
&= sup_{\|h\|=1} \left\langle \pi_{2|Tx-Ty|} h, h \right\rangle \\
&= sup_{\|h\|=1} \langle 2\, |Tx - Ty|\, h, h\rangle \\
&= sup_{\|h\|=1} \int_E (2\, |Tx - Ty|\, h)(t) \cdot \overline{h(t)} dt
\end{aligned}$$

$$\begin{aligned}
&\leq 2sup_{\|h\|=1} \int_E \left[\int_E |K(t, s, x(s) - K(t, s, y(s))| \right] |h(t)|^2\, dt \\
&\leq 2sup_{\|h\|=1} \int_E \left[\int_E k\, |\phi(t, s)(x(s) - y(s))|\, ds \right] |h(t)|^2\, dt \\
&\leq 2ksup_{\|h\|=1} \int_E \left[\int_E |\phi(t, s)|\, ds \right] |h(t)|^2\, dt \cdot \|x - y\|_\infty \\
&\leq ksup_{t\in E} \int_E |\phi(t, s)|\, ds \cdot sup_{\|h\|=1} \int_E |h(t)|^2\, dt2\, \|x - y\|_\infty \\
&\leq 2k\, \|x - y\|_\infty \\
&= k\, \|2(x - y)\|_\infty \\
&= k\, \left\|\pi_{(|x-y|+|x-y|)}\right\| \\
&= \|A\|\, \|S(x, x, y)\|\, .
\end{aligned}$$

Since $\|A\| < 1$ and by Corollary 3.1, the integral equation has a unique solution in $L^\infty(E)$. □

Example 4.2. *Suppose that H is a Hilbert space, $L(H)$ is the set of linear bounded operators on H. Let $A_1, A_2, A_3, ...A_n, ... \in L(H)$, which satisfy $\sum_{n=1}^{\infty} \|A_n\|^2 < 1$ and $X \in L(H)$ and $Q \in L(H)_+$. Then the operator equation*

$$X - \sum_{n=1}^{\infty} A_n^* X A_n = Q$$

has a unique solution in $L(H)$.

Proof. Put $\alpha = \sum_{n=1}^{\infty} \|A_n\|^2$. If $\alpha = 0$, then the equation has a unique solution in $L(H)$. Assume that $\alpha > 0$. Choose a positive operator $T \in L(H)$. For every $X, Y \in L(H)$, set

$$S(X, Y, Z) = (\|X - Z\| + \|Y - Z\|)\, T.$$

It is easy to verify that $S(X, Y, Z)$ is a complete C^*-algebra valued S-metric. Consider the map $F : L(H) \to L(H)$ defined by

$$F(X) = \sum_{n=1}^{\infty} A_n^* X A_n + Q.$$

Then,

$$\begin{aligned} S(F(X), F(X), F(Y)) &= 2\,\|F(X) - F(Y)\|\, T \\ &= 2\left\|\sum_{n=1}^{\infty} A_n^*(X - Y)A_n\right\| T \\ &\preceq 2\sum_{n=1}^{\infty} \|A_n\|^2\, \|(X - Y)\|\, T \\ &= \alpha S(X, X, Y) \\ &= (\alpha^{\frac{1}{2}} I)^* S(X, X, Y)(\alpha^{\frac{1}{2}} I). \end{aligned}$$

Using Theorem 3.1, there exists a unique fixed point X in $L(H)$. Furthermore, since $\sum_{n=1}^{\infty} A_n^* X A_n + Q$ is a positive operator, the solution is a Hermitian operator. □

As a special case of Example 4.2, one can consider the following matrix equation, which can also be found in [14]:

$$X - \sum_{n=1}^{m} A_n^* X A_n = Q$$

where Q is a positive definite matrix and $A_1, A_2, \cdots A_m$ are arbitrary $n \times n$ matrices with $\sum_{n=1}^{m} \|A_n\| < 1$. Using Example 4.2, there exists a unique Hermitian matrix solution.

Remark 4.1. *The step function is an example of an integrable function but not a continuous function. In Example 4.1, some special cases of a step function, which satisfies condition (2), have been mentioned.*

5. **Conclusions**

In this paper, we extend S-metric space [12] in the setting of C^*- algebra-valued S-metric space and prove some fixed point theorems for self mappings with contactive conditions. Fixed point theorems for operators in metric space are widely investigated and have found various applications in differential and integral equations [1, 5]. The important applications of our results are existence and uniqueness of a solution of integral equation and operator equation, which will be obtained in this paper.

6. **Future Work**

The continuation of this research is considering C^*-algebra-valued S_b-metrics. We introduce the notion of C^*-algebra-valued S_b-metric space, as a generalization of C^*-algebra-valued S-metric space. metric space.

Definition 6.1. *Let X be a nonempty set and $b \in \mathcal{A}'$ such that $b \preceq 1_{\mathcal{A}}$. Suppose the mapping $S : X \times X \times X \to \mathcal{A}$ satisfies:*
(1) $0_{\mathcal{A}} \preceq S(x, y, z)$ *for all* $x, y, z \in X$;
(2) $S(x, y, z) = 0_{\mathcal{A}}$ *if and only if* $x = y = z$;
(3) $S(x, y, z) \preceq b[S(x, x, a) + S(y, y, a) + S(z, z, a)]$ *for all* $x, y, z, a \in X$.

Then S is said to be C^*- algebra-valued S_b-metric on X and $(X, \mathcal{A}, S)$ is said to be a C^*-algebra-valued S_b-metric space.

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