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MULTIPLIERS AND SAMPLING THEORY FOR CONTINUOUS FRAMES IN HILBERT SPACES

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In this paper, we discuss a norm of continuous Bessel multipliers $M_{m,F,G}$, $m \in L^p(\Omega)$, $1 \le p \le \infty$. In particular, we show that there exists a unique continuous Bessel multiplier operator $M_{m,F,G}$, for $m \in L^p(\Omega)$, $1 < p < \infty$, and it is bounded linear after defining continuous Bessel multipliers $M_{m,F,G}$ for $m \in L^1(\Omega) \cup L^\infty(\Omega)$. Finally, sampling theory for continuous frames is discussed and each signal in the range of the analysis operator can be reconstructed in terms of its sampled values.

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1. Introduction and Preliminaries

Frames were first introduced in 1952 by Duffin and Schaeffer [9]. Frames have very important and interesting properties that make them very useful in sampling theory [10] and many other fields. A discrete frame is a countable family of elements in a separable Hilbert spaces, which allows a stable not necessarily unique decomposition of arbitrary elements into expansions of frame elements [8]. Let $\mathcal{H}$ be a separable Hilbert space. A sequence $\{f_j\}_{j\in\mathbb{N}} \subseteq \mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A||f||^2 \le \sum_{j\in\mathbb{N}} |\langle f, f_j\rangle|^2 \le B||f||^2, \quad (f \in \mathcal{H}).$$

Later, the theory of frames has been generalized in different ways by many authors. The concept of a generalization of frames to a family indexed by some locally compact space endowed with a Radon measure was proposed by G. Kaiser [12] and independently by Ali, Antoine and Gazeau [1]. These frames are known as continuous frames. The continuous wavelet transformation and short time Fourier transformation are examples of continuous frames. Let $\mathcal{H}$ be a separable Hilbert space and (Ω, μ) be a measure space. A mapping $F : \Omega \to \mathcal{H}$ is called a continuous frame if the mapping $\omega \mapsto \langle F(\omega), f\rangle$ is measurable for all $f \in \mathcal{H}$ and there exist constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \int_\Omega |\langle f, F(\omega)\rangle|^2 \, d\mu(\omega) \le B\|f\|^2, \quad (f \in \mathcal{H}). \tag{1}$$

The constants A and B are called continuous frame bounds. F is called a tight continuous frame if $A = B$ and it is called a Parseval continuous frame if $A = B = 1$. The mapping F is called Bessel mapping or shorter Bessel if only the righthand inequality in (1) holds.

Let F be a Bessel mapping for $\mathcal{H}$ with respect to (Ω, μ). Then the operator $T_F : \mathcal{H} \to L^2(\Omega)$ defined by

$$(T_F f)(\omega) = \langle f, F(\omega)\rangle, \quad (f \in \mathcal{H}, \ \omega \in \Omega),$$

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is well defined, linear and bounded. It is called an analysis operator of F and its adjoint is given by

$$\langle T_F^*\phi, f\rangle = \int_\Omega \phi(\omega)\langle F(\omega), f\rangle \; d\mu(\omega), \quad (\phi \in L^2(\Omega), \; f \in \mathcal{H}).$$

Now, suppose that F is a continuous frame with frame bounds A and B, respectively. We can define the operator $S_F = T_F^* T_F$. Since $\langle S_F f, f\rangle = \langle T_F^* T_F f, f\rangle = \int_\Omega |\langle f, F(\omega)\rangle|^2 \; d\mu(\omega)$, we see that S_F is positive and $AI \le S_F \le BI$. Hence, S_F is invertible. We call S_F a continuous frame operator of F and use the notation $S_F f = \int_\Omega \langle f, F(\omega)\rangle F(\omega) \; d\mu(\omega)$. Thus, every $f \in \mathcal{H}$ has the representations

$$f = S_F^{-1} S_F f = S_F^{-1} \int_\Omega \langle f, F(\omega)\rangle F(\omega) \; d\mu(\omega) = \int_\Omega \langle f, F(\omega)\rangle S_F^{-1} F(\omega) \; d\mu(\omega),$$

$$f = S_F S_F^{-1} f = \int_\Omega \langle S_F^{-1} f, F(\omega)\rangle F(\omega) \; d\mu(\omega) = \int_\Omega \langle f, S_F^{-1} F(\omega)\rangle F(\omega) \; d\mu(\omega).$$

For more details one can see [8].

It is well-known that discrete Bessel sequences in a Hilbert space are norm bounded above but this is not true for any continuous Bessel mappings (see [6]).

The following is a well known example in wavelet frame that is a continuous frame.

Example 1.1. *Let $g \in L^2(\mathbb{R})$ be an admissible function, i.e., $C_g := \int_{-\infty}^{+\infty} \frac{|\widehat{g}(\xi)|^2}{|\xi|} \; d\xi < +\infty$, and for $a, b \in \mathbb{R}$, $a \neq 0$,*

$$g_{a,b}(x) = \frac{1}{\sqrt{|a|}} g(\frac{x-b}{a}), \quad (x \in \mathbb{R}),$$

then $\{g_{a,b}\}_{a\neq 0, b\in\mathbb{R}}$ is a continuous frame for $L^2(\mathbb{R})$ with respect to $\mathbb{R}\backslash\{0\} \times \mathbb{R}$ equipped with the measure $\frac{db \; da}{a^2}$ and, for all $f \in L^2(\mathbb{R})$,

$$f = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W_g(f)(a,b) g_{a,b} \frac{db \; da}{a^2},$$

where W_g is the continuous wavelet transform defined by

$$W_g(f)(a,b) := \int_{-\infty}^{+\infty} f(x) \frac{1}{\sqrt{|a|}} \overline{g(\frac{x-b}{a})} dx.$$

For details, see the Proposition 11.1.1 and Corollary 11.1.2 in [8].

Next, Bessel multipliers are investigated by Peter Balazs [3, 4, 5] for Hilbert spaces. For Bessel sequences, the investigation of the operator $M = \sum m_j \langle f, f_j\rangle g_j$, where the analysis coefficients $\langle f, f_j\rangle$ are multiplied by a fix symbol $\{m_j\}$ before synthesis (with $\{g_j\}$), is very natural, useful and there are numerous applications of this kind of operators.

The paper is organized as follows. In Section 2, we will define the concept of continuous Bessel multipliers $M_{m,F,G}$ for $m \in L^1(\Omega) \cup L^\infty(\Omega)$, and we discuss the upper bound of multiplier operators for continuous frames. Besides, for $m \in L^p(\Omega)$, $1 < p < \infty$, we show that there exists a unique bounded linear operator $M_{m,F,G}$. In Section 3, sampling theory for continuous frames is discussed.

Throughout this paper, $\mathcal{H}$ will be separable Hilbert spaces.

2. **Multipliers for Continuous Frames**

Gabor multipliers [11] led to the introduction of Bessel and frame multipliers for Hilbert spaces. These operators are defined by a fixed multiplication pattern (the symbol) which is inserted between the analysis and synthesis operators [3, 4, 5].

In this section, we discuss a norm of continuous Bessel multipliers $M_{m,F,G}$ for $m \in L^1(\Omega) \cup L^\infty(\Omega)$. Next, for $m \in L^p(\Omega)$, $1 < p < \infty$, we show that there exists a unique bounded linear operator $M_{m,F,G}$. First, we give the following definition [6].

Throughout this section, let $B(\mathcal{K})$ be the set of all bounded linear operators from the Hilbert space $\mathcal{K}$ to $\mathcal{K}$.

Definition 2.1. *Let F and G be Bessel mappings for $\mathcal{H}$ with respect to (Ω, μ) and $m \in L^1(\Omega) \cup L^\infty(\Omega)$ be a measurable function. The operator $M_{m,F,G} : \mathcal{H} \to \mathcal{H}$ weakly defined by*

$$\langle M_{m,F,G} f, g\rangle = \int_\Omega m(\omega)\langle f, F(\omega)\rangle\langle G(\omega), g\rangle d\mu(\omega), \tag{2}$$

for all $f, g \in \mathcal{H}$, is called continuous Bessel multiplier of F and G with respect to the mapping m.

Proposition 2.1. [6, Theorem 3.10] *Let F and G be Bessel mappings for $\mathcal{H}$ with respect to (Ω, μ) and norm bounded with norm bounds L_F and L_G, respectively, i.e. $\|F(\omega)\| \le L_F$ and $\|G(\omega)\| \le L_G$ for almost every $\omega \in \Omega$. Let $m \in L^1(\Omega)$. Then the continuous Bessel multiplier $M_{m,F,G}$ is a well defined bounded linear operator and $\|M_{m,F,G}\| \le L_F L_G \|m\|_{L^1(\Omega)}$.*

Proposition 2.2. [6, Lemma 3.3] *Let F and G be Bessel mappings for $\mathcal{H}$ with respect to (Ω, μ) with bounds B_F and B_G. Let $m \in L^\infty(\Omega)$. Then the continuous Bessel multiplier $M_{m,F,G}$ is a well defined bounded linear operator and*

$$\|M_{m,F,G}\| \le \sqrt{B_F B_G}\|m\|_{L^\infty(\Omega)}.$$

We can now associate a continuous Bessel multiplier $M_{m,F,G} : \mathcal{H} \to \mathcal{H}$ to every function $m \in L^p(\Omega)$, $1 < p < \infty$, and prove that $M_{m,F,G}$ is a bounded linear operator. To prove this result we need a recall of the Riesz-Thorin interpolation theorem [7].

Theorem 2.1. *Let (Ω, μ) be a measurable space and (Ψ, ν) a σ-finite measure space. Let T be a linear transformation with the domain D consisting of all simple functions H on Ω such that $\mu\{\omega \in \Omega : H(\omega) \neq 0\} < \infty$, and such that the range of T is contained in the set of all measurable functions on Ψ. Suppose that $\alpha_1, \alpha_2, \beta_1$ and β_2 are numbers in $[0, 1]$ and there exist positive constants γ_1 and γ_2 such that*

$$\|TH\|_{L^{\frac{1}{\beta_j}}(\Psi)} \le \gamma_j \|f\|_{L^{\frac{1}{\alpha_j}}(\Omega)}, \quad j = 1, 2, \ (f \in D).$$

Then for $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_1 + \theta\alpha_2$ and $\beta = (1-\theta)\beta_1 + \theta\beta_2$, we have

$$\|TH\|_{L^{\frac{1}{\beta}}(\Psi)} \le \gamma_1^{1-\theta}\gamma_2^{\theta}\|f\|_{L^{\frac{1}{\alpha}}(\Omega)}, \quad (f \in D).$$

Now, we show that there exists a unique bounded linear operator $M_{m,F,G}$ for $m \in L^p(\Omega)$, $1 < p < \infty$.

Theorem 2.2. *Let F and G be Bessel mappings for $\mathcal{H}$ with respect to (Ω, μ) with bounds B_F and B_G and norm bounded with norm bounds L_F and L_G, respectively. Let $m \in L^p(\Omega)$, $1 < p < \infty$. Then there exists a unique bounded linear operator $M_{m,F,G} : \mathcal{H} \to \mathcal{H}$ such that*

$$\|M_{m,F,G}\| \le (L_F L_G)^{\frac{1}{p}} (B_F B_G)^{\frac{1}{2q}} \|m\|_{L^p(\Omega)},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $M_{m,F,G}$ is given by (2) for all f and g in $\mathcal{H}$ and all simple functions m on Ω for which

$$\mu\{\omega \in \Omega : m(\omega) \neq 0\} < \infty.$$

Proof. Let $L : \mathcal{H} \to L^2(\mathbb{R}^n)$ be a unitary operator between $\mathcal{H}$ and $L^2(\mathbb{R}^n)$. Let $m \in L^1(\Omega)$. Then, by Proposition 2.1, the linear operator $\widetilde{M}_{m,F,G} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, defined by $\widetilde{M}_{m,F,G} = L M_{m,F,G} L^{-1}$, is bounded and $\|\widetilde{M}_{m,F,G}\| \le L_F L_G \|m\|_{L^1(\Omega)}$.

Now, let $m \in L^{\infty}(\Omega)$, then by Proposition 2.2, the linear operator $\widetilde{M}_{m,F,G}$ is also bounded and

$$\|\widetilde{M}_{m,F,G}\| \leq \sqrt{B_F B_G}\|m\|_{L^{\infty}(\Omega)}.$$

Let D be the set of all simple functions m on Ω such that

$$\mu\{\omega \in \Omega : m(\omega) \neq 0\} < \infty.$$

Let $f \in L^2(\mathbb{R}^n)$ and T be the linear transformation from D into the set of all Lebesgue functions on $\mathbb{R}^n$ defined by

$$Tm = \widetilde{M}_{m,F,G} f, \quad (m \in D).$$

Then we have

$$\|Tm\|_{L^2(\mathbb{R}^n)} \leq L_F L_G \|m\|_{L^1(\Omega)} \|f\|_{L^2(\mathbb{R}^n)},$$

and

$$\|Tm\|_{L^2(\mathbb{R}^n)} \leq \sqrt{B_F B_G}\|m\|_{L^{\infty}(\Omega)} \|f\|_{L^2(\mathbb{R}^n)},$$

for all functions m in D. Thus, by Theorem 2.1, if we set $\theta = \frac{1}{q}$, then we get

$$\|Tm\|_{L^2(\mathbb{R}^n)} \leq (L_F L_G)^{\frac{1}{p}} (B_F B_G)^{\frac{1}{2q}} \|m\|_{L^p(\Omega)} \|f\|_{L^2(\mathbb{R}^n)}, \quad (m \in D).$$

Note that $\beta_1 = \beta_2 = \frac{1}{2}, \beta = \frac{1}{2}, \alpha_1 = 1, \alpha_2 = 0, \alpha = \frac{1}{p}, \gamma_1 = L_F L_G \|f\|_{L^2(\mathbb{R}^n)}$ and $\gamma_2 = \sqrt{B_F B_G}\|f\|_{L^2(\mathbb{R}^n)}$. Hence

$$\|\widetilde{M}_{m,F,G} f\|_{L^2(\mathbb{R}^n)} \leq (L_F L_G)^{\frac{1}{p}} (B_F B_G)^{\frac{1}{2q}} \|m\|_{L^p(\Omega)} \|f\|_2, \quad (m \in D).$$

Therefore

$$\|\widetilde{M}_{m,F,G}\| \leq (L_F L_G)^{\frac{1}{p}} (B_F B_G)^{\frac{1}{2q}} \|m\|_{L^p(\Omega)}, \quad (m \in D).$$

Let $m \in L^p(\Omega)$, $1 < p < \infty$. Then there exists a sequence $\{m_j\}_{j=1}^{\infty}$ of functions in D such that $m_j \to m$ in $L^p(\Omega)$ as $j \to \infty$. Then

$$\|\widetilde{M}_{m_i,F,G} - \widetilde{M}_{m_j,F,G}\| \leq (L_F L_G)^{\frac{1}{p}} (B_F B_G)^{\frac{1}{2q}} \|m_i - m_j\|_p \to 0,$$

as $i, j \to \infty$. Therefore, $\{\widetilde{M}_{m_j,F,G}\}_{j=1}^{\infty}$ is a Cauchy sequence in $B(L^2(\mathbb{R}^n))$. Using the completeness of $B(L^2(\mathbb{R}^n))$, we can find a bounded linear operator $\widetilde{M}_{m,F,G} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that $\widetilde{M}_{m_j,F,G} \to M_{m,F,G}$ as $j \to \infty$. So

$$\|M_{m,F,G}\| \leq (L_F L_G)^{\frac{1}{p}} (B_F B_G)^{\frac{1}{2q}} \|m\|_{L^p(\Omega)}, (m \in D),$$

and the operator $M_{m,F,G} = L^{-1}\widetilde{M}_{m,F,G} L \in B(\mathcal{H})$ is bounded linear and satisfying the conclusions of the theorem if $m \in L^p(\Omega)$, $1 < p < \infty$.

To prove uniqueness, let $m \in L^p(\Omega)$, $1 < p < \infty$, and suppose that $P_m : \mathcal{H} \to \mathcal{H}$ is another bounded linear operator satisfying the conclusions of the theorem. Let $Q : L^p(\Omega) \to B(\mathcal{H})$ be the linear operator defined by

$$Qm = M_{m,F,G} - P_m, \quad (m \in L^p(\Omega)).$$

Then $\|Qm\| \leq 2\,(L_F L_G)^{\frac{1}{p}} (B_F B_G)^{\frac{1}{2q}} \|m\|_{L^p(\Omega)}, \quad (m \in L^p(\Omega))$. Since $M_{m,F,G}$ and P_m are bounded linear operators satisfying the conclusions of the theorem, in particular, they are given by (2) for all f and g in $\mathcal{H}$ and all simple functions m on Ω for which $\mu\{\omega \in \Omega : m(\omega) \neq 0\} < \infty$, so $Qm = M_{m,F,G} - P_m$ is equal to the zero operator on $\mathcal{H}$ for all $m \in D$. Thus, $Q : L^p(\Omega) \to B(\mathcal{H})$ is a bounded linear operator that is equal to zero on the dense subspace D of $L^p(\Omega)$. Therefore $P_m = M_{m,F,G}$ for all functions m in $L^p(\Omega)$. □

3. Sampling Theory for Continuous Frames

In this section, we show that the range of the analysis operator of a continuous frame is a reproducing kernel Hilbert space and in particular it is closed. Recall that a Hilbert space $\mathcal{H}$ of complex-valued functions on a set Ω is called a reproducing kernel Hilbert space if the evaluation functionals $E_z(f) = f(z),\ z \in \Omega,\ f \in \mathcal{H}$, are bounded linear functionals (see [2] for more details). Moreover, each signal in the range of the analysis operator can be reconstructed in terms of its sampled values.

Throughout this section, by $R(T_F)$ we always denote the range of the analysis operator T_F.

Theorem 3.1. *Let F be a continuous frame for $\mathcal{H}$ with respect to (Ω, μ). Then the range $R(T_F)$ of T_F is a reproducing kernel Hilbert space.*

Proof. First we show that $R(T_F)$ is a Hilbert space. It is enough to show that $R(T_F)$ is a closed subspace of $L^2(\Omega)$. Closeness of the range $R(T_F)$ of the analysis operator follows immediately from the fact that T_F is bounded from below.
Now let $m \in R(T_F)$. Then there exists $f \in \mathcal{H}$ such that $m = T_F f$. For $\omega \in \Omega$ we get

$$\begin{aligned}(T_F f)(\omega) = \langle f, F(\omega)\rangle &= \int_\Omega \langle f, F(\xi)\rangle \langle S_F^{-1} F(\xi), F(\omega)\rangle d\mu(\xi)\\ &= \int_\Omega (T_F f)(\xi) \langle S_F^{-1} F(\xi), F(\omega)\rangle d\mu(\xi) = \int_\Omega m(\xi) \langle S_F^{-1} F(\xi), F(\omega)\rangle d\mu(\xi),\end{aligned}$$

that is, $m(\omega) = \int_\Omega m(\xi)\langle S_F^{-1}F(\xi), F(\omega)\rangle d\mu(\xi)$, which implies that $R(T_F)$ is a reproducing kernel Hilbert space with reproducing kernel

$$\mathcal{P}^\omega_{S_F^{-1}F}(\xi) := \langle S_F^{-1} F(\xi), F(\omega)\rangle.$$

□

Remark 3.1. *In Theorem 3.1, suppose that F is a Parseval continuous frame. Then $R(T_F)$ is the reproducing kernel Hilbert space with the reproducing kernel $\mathcal{P}^\omega_F(\xi) := \langle F(\xi), F(\omega)\rangle$.*

Theorem 3.2. *Suppose that F is a Parseval continuous frame for $\mathcal{H}$ with respect to (Ω, μ). Let $\{\omega_j\}_{j=1}^\infty$ be a countable family in Ω such that $\{F(\omega_j)\}_{j=1}^\infty$ is an orthonormal basis for $\mathcal{H}$. Then for $m \in R(T_F)$ we get $m(\omega) = \sum_{j=1}^\infty m(\omega_j)\mathcal{K}_F^{\omega_j}(\omega)$, $(\omega \in \Omega)$, where $\mathcal{K}_F^{\omega_j} := T_F F(\omega_j)$.*

Proof. Since T_F is an isometry operator, $\{\mathcal{K}_F^{\omega_j}\}_{j=1}^\infty = \{T_F F(\omega_j)\}_{j=1}^\infty$ is an orthonormal basis for the range $R(T_F)$ of T_F. So, if $m \in R(T_F)$, then $m = \sum_{j=1}^\infty \langle m, \mathcal{K}_F^{\omega_j}\rangle \mathcal{K}_F^{\omega_j}$, the series being absolutely convergent in $L^2(\Omega)$. In fact, for every $\omega \in \Omega$ we get

$$\sum_{j=1}^\infty |\langle m, \mathcal{K}_F^{\omega_j}\rangle||\mathcal{K}_F^{\omega_j}(\omega)| \le \left(\sum_{j=1}^\infty |\langle m, \mathcal{K}_F^{\omega_j}\rangle|^2\right)^{\frac12} \left(\sum_{j=1}^\infty |\mathcal{K}_F^{\omega_j}(\omega)|^2\right)^{\frac12} = \|m\|_{L^2(\Omega)} \left(\sum_{j=1}^\infty |\mathcal{K}_F^{\omega_j}(\omega)|^2\right)^{\frac12}.$$

By Theorem 3.1, $R(T_F)$ is a reproducing kernel Hilbert space. Since $\mathcal{K}_F^{\omega_j} \in R(T_F)$, $j = 1, 2, \cdots$, it follows that $\mathcal{K}_F^{\omega_j}(\omega) = \int_\Omega \mathcal{K}_F^{\omega_j}(\xi)\mathcal{P}_F^\omega(\xi) d\mu(\xi) = \langle \mathcal{K}_F^{\omega_j}, \overline{\mathcal{P}_F^\omega}\rangle_{L^2(\Omega)}$, $(\omega \in \Omega)$, for $j = 1, 2, \cdots$. So we have

$$\begin{aligned}\sum_{j=1}^\infty |\langle m, \mathcal{K}_F^{\omega_j}\rangle||\mathcal{K}_F^{\omega_j}(\omega)| &\le \|m\|_{L^2(\Omega)} \left(\sum_{j=1}^\infty |\mathcal{K}_F^{\omega_j}(\omega)|^2\right)^{\frac12}\\ &= \|m\|_{L^2(\Omega)} \left(\sum_{j=1}^\infty |\langle \mathcal{K}_F^{\omega_j}, \overline{\mathcal{P}_F^\omega}\rangle|^2\right)^{\frac12} = \|m\|_{L^2(\Omega)} \|\mathcal{P}_F^\omega\|_{L^2(\Omega)}.\end{aligned}$$

Thus the absolute convergence says that

$$m(\omega) = \sum_{j=1}^{\infty} \langle m, \mathcal{K}_F^{\omega_j} \rangle \mathcal{K}_F^{\omega_j}(\omega) = \sum_{j=1}^{\infty} \left(\int_{\Omega} m(\xi) \overline{\mathcal{K}_F^{\omega_j}(\xi)} d\mu(\xi) \right) \mathcal{K}_F^{\omega_j}(\omega), \tag{3}$$

for all ω in Ω. Also, for all ξ in Ω we get

$$\mathcal{K}_F^{\omega_j}(\xi) = (T_F F(\omega_j))(\xi) = \langle F(\omega_j), F(\xi) \rangle = \overline{\langle F(\xi), F(\omega_j) \rangle} = \overline{\mathcal{P}_F^{\omega_j}(\xi)}, \tag{4}$$

for $j = 1, 2, \cdots$. So we have

$$\begin{aligned} m(\omega) &= \sum_{j=1}^{\infty} \left(\int_{\Omega} m(\xi) \overline{\mathcal{K}_F^{\omega_j}(\xi)} d\mu(\xi) \right) \mathcal{K}_F^{\omega_j}(\omega) && \text{(by Eq. 3)} \\ &= \sum_{j=1}^{\infty} \left(\int_{\Omega} m(\xi) \mathcal{P}_F^{\omega_j}(\xi) d\mu(\xi) \right) \mathcal{K}_F^{\omega_j}(\omega) && \text{(by Eq. 4)} \\ &= \sum_{j=1}^{\infty} m(\omega_j) \mathcal{K}_F^{\omega_j}(\omega), && \text{(by Remark 3.1)} \end{aligned}$$

for all ω in Ω. Thus, $m(\omega) = \sum_{j=1}^{\infty} m(\omega_j) \mathcal{K}_F^{\omega_j}(\omega)$. □

Remark 3.2. *In Theorem 3.2, each signal m processed by means of the analysis operator $T_F : \mathcal{H} \to L^2(\Omega)$ can be reconstructed in terms of its sampled values $\{m(\omega_j\}_{j=1}^{\infty}$ on Ω.*

REFERENCES

[1] *S. T. Ali, J. P. Antoine and J. P. Gazeau*, Continuous frames in Hilbert space, Ann. Physics, **222**(1993), No. 1, 1–37.

[2] *N. Aronszajn*, Theory of reproducing kernels, Trans. Am. Math. Soc., **68**(1950), 337–404.

[3] *P. Balazs*, Basic Definition and Properties of Bessel Multipliers, J. Math. Anal. Appl., **325**(2007), No. 1, 571–585.

[4] *P. Balazs*, Hilbert-Schmidt Operators and Frames - Classifications, Approximation by Multipliers and Algorithms, Int. J. Wavelets Multiresult. Inf. Process, **6**(2008), No. 2, 315–330.

[5] *P. Balazs*, Matrix representations of operators using frames, Sampl. Theory Signal Image Process., **7**(2008), No. 1, 39–54.

[6] *P. Balazs, D. Bayer and A. Rahimi*, Multipliers for continuous frames in Hilbert spaces, J. Phys. A: Math. Theor., **45**(2002), 244023.

[7] *J. Bergh and J. Lofstrom*, Interpolation Spaces, An Introduction, Springer-Hill, 1964.

[8] *O. Christensen*, An Introduction to Frames and Riesz Bases, Brikhouser, 2003.

[9] *J. Duffin and A. C. Schaeffer*, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc., **72**(1952), 341–366.

[10] *Y. C. Eldar and T. Werther*, General framework for consistent sampling in Hilbert spaces, Int. J. Wavelets Multi. Inf. Process., **3**(2005), No. 3, 347–359.

[11] *H. G. Feichtinger and K. Nowak*, A First Survey of Gabor Multipliers, H. G. Feichtinger and T. Strohmer, Ch. 5, pp. 9–128, 2003.

[12] *G. Kaiser*, A Friendly Guide to Wavelets, Birkhäuser, Boston, 1994.