# A NOTE ON THE STABILITY ANALYSIS OF A CLASS OF NONLINEAR SYSTEMS - AN LMI APPROACH

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*The main goal of the paper is to study the equilibria of a nonlinear system, proving the existence and uniqueness of an equilibrium point in the positive ortant. We also provide numerically tractable conditions (by using Linear Matrix Inequalities techniques) to check the asymptotic stability of the equilibrium point. An illustrative numerical example is closing the paper along with some conclusions.*

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## 1. Introduction

Consider the following nonlinear system

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t} &= b - dX - \left(\sum_{j=1}^{M} k_j^+ Y_j\right) X + \sum_{j=1}^{M} k_j^- Z_j, \\ \frac{\mathrm{d}Y_j}{\mathrm{d}t} &= \beta_j - \delta_j Y_j - k_j^+ Y_j X + (k_j^- + K_j) Z_j, \quad j = \overline{1, M} \\ \frac{\mathrm{d}Z_j}{\mathrm{d}t} &= -(\sigma_j + k_j^- + K_j) Z_j + k_j^+ Y_j X, \quad j = \overline{1, M} \end{aligned} \tag{1}$$

Here $X$, $Y_j$, $Z_j$ ($j = \overline{1, M}$) denote concentrations in an enzymatic reaction, which appears in the study of micro RNA - messenger RNA dynamics (for further details see [5], [2], [7], [4], [10], [6]). The coefficients $b, d, \beta_j, \delta_j$, $k_j^+, k_j^-, K_j$ and $\sigma_j$ are all positive.

In this paper we study the positive ortant equilibria of the system, proving existence and uniqueness (Theorem 2.1). Further, in Section 3, we provide numerically tractable conditions (by using Linear Matrix Inequalities techniques), to check the asymptotic stability of the equilibrium point. An illustrative numerical example is closing the paper along with some conclusions.

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## 2. **Problem statement**

The system can be rewritten as

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = F(\Phi), \quad \text{where } \Phi := \begin{bmatrix} X \\ Y_1 \\ \vdots \\ Y_M \\ Z_1 \\ \vdots \\ Z_M \end{bmatrix}$$

and $F$ is the appropriate vector field defined on $\mathbb{R}^{2M+1}$ and associated to the system (1). For every $\Phi_0 = (X^0, Y_i^0, Z_i^0) \in \mathbb{R}_+ \times \mathbb{R}_+^{2M}$, denote by $\Phi(t; t_0, \Phi_0)$ the solution of the Cauchy problem

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = F(\Phi), \quad \Phi(t_0) = \Phi_0.$$

**Remark 2.1.** *Obviously, the Existence and Uniqueness Theorem applies to the system (1); moreover, the positive ortant $\mathbb{R}_+^{2M+1}$ is a positively invariant set for the system - see* [8].

The first main result is:

**Theorem 2.1.** *For every positive set of parameters $b, \beta_j,\ d, \delta_j, \sigma_j,\ k_j^+, k_j^-, K_j,\ j = \overline{1, M}$, the system (1) has an unique equilibrium point $(\widetilde{X}, \widetilde{Y}_j, \widetilde{Z}_j)$ in $\mathbb{R}_+^{2M+1}$, $j = \overline{1, M}$; moreover*

$$X \in \left(0\,,\ \frac{b}{d}\right), \quad Y_j \in \left(\frac{\beta_j}{\delta_j + B_j \frac{b}{d}}\,,\ \frac{\beta_j}{\delta_j}\right),$$

$B_j = \frac{k_j^+ \sigma_j}{\sigma_j + k_j^- + K_j}$.

*Proof.* From (1) we get the following equilibria $2M+1$ algebraic equations

$$b - dX - \left(\sum_{j=1}^{M} k_j^+ Y_j\right) X + \sum_{j=1}^{M} k_j^- Z_j = 0, \tag{2}$$

$$\beta_j - \delta_j Y_j - k_j^+ Y_j X + (k_j^- + K_j) Z_j = 0, \quad j = \overline{1, M} \tag{3}$$

$$-(\sigma_j + k_j^- + K_j) Z_j + k_j^+ Y_j X = 0, \quad j = \overline{1, M} \tag{4}$$

From the last $M$ equations (4) we get

$$Z_j = \frac{k_j^+ Y_j X}{\sigma_j + k_j^- + K_j}, \quad j = \overline{1, M}.$$

Replace $Z_j$ in the first $M+1$ equations (2)-(3) and obtain

$$\begin{aligned} b - dX - \left( \sum_{j=1}^{M} \frac{k_j^+(K_j+\sigma_j)}{\sigma_j + k_j^- + K_j} Y_j \right) X &= 0 \\ \beta_j - \delta_j Y_j - \frac{k_j^+ \sigma_j}{\sigma_j + k_j^- + K_j} Y_j X &= 0, \ \ j = \overline{1, M} \end{aligned}$$

or, equivalently,

$$\begin{aligned} X \left( d + \sum_{j=1}^{M} \frac{k_j^+(K_j+\sigma_j)}{\sigma_j + k_j^- + K_j} Y_j \right) &= b \\ Y_j \left( \delta_j + \frac{k_j^+ \sigma_j}{\sigma_j + k_j^- + K_j} X \right) &= \beta_j, \ \ j = \overline{1, M}. \end{aligned}$$

With the notations below

$$\begin{aligned} A_j &= \frac{k_j^+(K_j+\sigma_j)}{\sigma_j + k_j^- + K_j} \\ B_j &= \frac{k_j^+ \sigma_j}{\sigma_j + k_j^- + K_j}, \end{aligned}$$

the previous system of equations becomes

$$X \left( d + \sum_{j=1}^{M} A_j Y_j \right) = b \tag{5}$$

$$Y_j \left( \delta_j + B_j m \right) = \beta_j, \ \ j = \overline{1, M}. \tag{6}$$

Take $Y_j = \dfrac{\beta_j}{\delta_j + B_j X}$, $j = \overline{1, M}$, and replace $Y_j$ in (5):

$$X \left( d + \sum_{j=1}^{M} A_j \frac{\beta_j}{\delta_j + B_j X} \right) = b.$$

Let $f : \mathbb{R}_+ \to \mathbb{R}$, $f(X) = X \left( d + \sum_{j=1}^{M} A_j \dfrac{\beta_j}{\delta_j + B_j X} \right)$.

Then $f'(X) = d + \sum_{j=1}^{M} \dfrac{A_j \beta_j \delta_j}{(\delta_j + B_j X)^2} > 0$, which shows that $f$ is increasing.

Noticing that $f(0) = 0 < b$ and $f(\frac{b}{d}) = \frac{b}{d} \left( d + \sum_{j=1}^{M} A_j \frac{\beta_j}{\delta_j + B_j \frac{b}{d}} \right) > b$, one gets that the equation $f(X) = b$ has a unique solution $\widetilde{X} \in \left(0\,,\,\frac{b}{d}\right)$. A straight computation shows that

$$\widetilde{Y}_j = \frac{\beta_j}{\delta_j + B_j \widetilde{X}} \in \left( \frac{\beta_j}{\delta_j + B_j \frac{b}{d}}\,,\, \frac{\beta_j}{\delta_j} \right).$$

□

## 3. LMI stability conditions

In order to investigate the asymptotic stability of the equilibrium point, we will use Lyapunov's stability theorem in first approximation and obtaining an LMI (Linear Matrix Inequality) sufficient condition.
Let us first translate the system (1) to the origin. Define the deviations with respect to the equilibrium point in Theorem 2.1 by

$$x = X - \widetilde{X},\ y_j = Y_j - \widetilde{Y}_j,\ z_j = Z_j - \widetilde{Z_j}, j = \overline{1, M},$$

$x = X - \widetilde{X}$, $y = Y - \widetilde{Y}$ and $z = Z - \widetilde{Z}$, respectively. Then, with this change of variables, the dynamics of the deviations' system are given by

$$\begin{aligned}
\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} &= -\left(d + \sum_{j=1}^{M} k_j^+ \widetilde{Y}_j\right) x - \sum_{j=1}^{M} k_j^+ \widetilde{X}\, y_j + \sum_{j=1}^{M} k_j^- z_j - \sum_{j=1}^{M} k_j^+ x y_j, \\
\dot{y}_j = \frac{\mathrm{d}y_j}{\mathrm{d}t} &= -k_j^+ \widetilde{Y}_j x - (\delta_j + k_j^+ \widetilde{X})\, y_j + (k_j^- + K_j) z_j - k_j^+ x y_j, \quad j = \overline{1, M} \qquad (7) \\
\dot{z}_j = \frac{\mathrm{d}z_j}{\mathrm{d}t} &= k_j^+ \widetilde{Y}_j x + k_j^+ \widetilde{X} y_j - (\sigma_j + k_j^- + K_j) z_j + k_j^+ x y_j, \quad j = \overline{1, M}
\end{aligned}$$

Obviously the origin is an equilibrium point for (7), exhibiting the same stability and topological properties as $(\widetilde{X}, \widetilde{Y}_j, \widetilde{Z_j})$ for the system (1) - see [9], Ch.4.

The translated system (7) rewrites now as

$$\left[\begin{array}{c} \dot{x} \\ \hline \dot{y}_1 \\ \vdots \\ \dot{y}_M \\ \hline \dot{z}_1 \\ \vdots \\ \dot{z}_M \end{array}\right] = \left[\begin{array}{c|c|c} -(d + \sum_{j=1}^{M} k_j^+ \widetilde{Y}_j) & -\begin{bmatrix} k_1^+ & \dots & k_M^+ \end{bmatrix} \widetilde{X} & \begin{bmatrix} k_1^- & \dots & k_M^- \end{bmatrix} \\ \hline -\begin{bmatrix} k_1^+ \widetilde{Y}_1 \\ \vdots \\ k_M^+ \widetilde{Y}_M \end{bmatrix} & -\operatorname{diag}(\delta_j + k_j^+ \widetilde{X}) & \operatorname{diag}(k_j^+ + K_j) \\ \hline \begin{bmatrix} k_1^+ \widetilde{Y}_1 \\ \vdots \\ k_M^+ \widetilde{Y}_M \end{bmatrix} & \operatorname{diag}(k_j^+ \widetilde{X}) & -\operatorname{diag}(\sigma_j + k_j^- + K_j) \end{array}\right] \left[\begin{array}{c} x \\ \hline y_1 \\ \vdots \\ y_M \\ \hline z_1 \\ \vdots \\ z_M \end{array}\right]$$

$$+ \left[\begin{array}{c} -\sum_{j=1}^{M} k_j^+ x y_j \\ \hline -k_1^+ x y_1 \\ \vdots \\ -k_M^+ x y_M \\ \hline k_1^+ x y_1 \\ \vdots \\ k_M^+ x y_M \end{array}\right]$$

or, equivalently,

$$\dot{\xi} = A\xi + g(\xi). \tag{8}$$

Here $\xi^T = \begin{bmatrix} x & y_1 & \dots & y_M & z_1 & \dots & z_M \end{bmatrix}$, $A = A_0 + \sum_{j=1}^{M} A_j$, $g(\xi) = \sum_{j=1}^{M} a_j y_j x$,

$$A_0 = \left[\begin{array}{c|c|c} -d & O & \begin{bmatrix} k_1^- & \dots & k_M^- \end{bmatrix} \\ \hline O & -\operatorname{diag}\delta_j & \operatorname{diag}(k_j^+ + K_j) \\ \hline O & O & -\operatorname{diag}(\sigma_j + k_j^- + K_j) \end{array}\right], \quad A_j = a_j v_j^T,$$

$$a_j = k_j^+ \left[\begin{array}{c} -1 \\ \hline 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array}\right] \quad \text{and} \quad v_j = \left[\begin{array}{c} \widetilde{Y}_j \\ \hline 0 \\ \vdots \\ 0 \\ \widetilde{X} \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{array}\right].$$

Then the Jacobian matrix associated to the system (7) in a point $\xi \in \mathbb{R}^{2M+1}$ is

$$J(\xi) = A + \sum_{j=1}^{M} a_j (y_j + x),$$

hence

$$J(0) = A = A_0 + \sum_{j=1}^{M} A_j. \tag{9}$$

**Remark 3.1.** *The Jacobian matrix associated to the system (7) does not depend on $\widetilde{Z}_j$. In order to investigate the stability of the origin for the "translated" system, we will make use of Lyapunov first Theorem. For proving that the origin is an asymptotically stable equilibrium point for (7), it is sufficient to check that $J(0)$ is a Hurwitz matrix, or equivalently, there exists a symmetric positive definite matrix $P$ such that*

$$A^T P + PA < 0 \Longleftrightarrow A_0^T P + PA_0 + \sum_{j=1}^{M} v_j p_j^T + \sum_{j=1}^{M} p_j v_j^T < 0, \;\; p_j = Pa_j. \tag{10}$$

*One can show that*

$$v_j p_j^T + p_j v_j^T < v_j v_j^T + p_j p_j^T \leq \lambda_{max}(v_j v_j^T) I_{2M+1} + P a_j a_j^T P.$$

*Since*

$$\lambda_{max}(v_j v_j^T) = \widetilde{X}^2 + \widetilde{Y}_j^2 \leq \frac{b^2}{d^2} + \frac{\beta_j^2}{\delta_j^2},$$

*it follows that the LMI (10) is satisfied whenever the following (Riccati) matrix inequality*

$$A_0^T P + PA_0 + \sum_{j=1}^{M} \left( \frac{b^2}{d^2} + \frac{\beta_j^2}{\delta_j^2} \right) I + PBB^T P < 0, \quad \text{where } B = \begin{bmatrix} a_1 & a_2 & \dots & a_M \end{bmatrix} \tag{11}$$

*holds. Equivalently, by using a Schur complement argument and denoting by* $\rho = \sum_{j=1}^{M} \left( \frac{b^2}{d^2} + \frac{\beta_j^2}{\delta_j^2} \right) > 0$, *the above inequality becomes*

$$\begin{bmatrix} A_0^T P + PA_0 + \rho I_{2M+1} & PB \\ B^T P & -I_M \end{bmatrix} < 0. \tag{12}$$

From the above considerations the next important result follows.

**Proposition 3.1.** *If there exists a symmetric positive definite matrix $P$ satisfying the above LMI (12), then the origin is an asymptotically stable equilibrium point for the translated system (8).*

This last relation is an LMI in the unknown $P$ and can be solved by using existing semidefinite programming software packages.

As we will show in the next section, we have used the *cvx* programming environment developed by Boyd *et. al* [3] and run the SDPT3 semidefinite programming package.

## 4. **Numerical examples**

Consider $M = 2$ and the following parameters (coefficients): $b = 4, \beta_1 = 1.5, \beta_2 = 0.1$; $d = 12, \delta_1 = 14, \delta_2 = 11$; $k_1^+ = 10, k_2^+ = 5$; $k_1^- = 3, k_2^- = 0.1$; $K_1 = 0.8, K_2 = 1$ and $\sigma_1 = 1.5, \sigma_2 = 10$.

In this case the feasibility problem (12) has a positive definite solution

$$P = \begin{bmatrix} 4.6996 & -2.9796 & -2.2656 & 2.0782 & 2.1960 \\ -2.9796 & 1.9734 & 1.4191 & -1.3454 & -1.3977 \\ -2.2656 & 1.4191 & 2.0419 & -1.0036 & -0.8466 \\ 2.0782 & -1.3454 & -1.0036 & 0.9881 & 0.9665 \\ 2.1960 & -1.3977 & -0.8466 & 0.9665 & 1.3968 \end{bmatrix},$$

and the spectrum of $P$ is $\Lambda_P = \{0.0475, 0.0625, 0.2566, 0.9745, 9.7586\}$. Furthermore, the spectrum of the left-hand side in (12) is

$$\Lambda = \{-242.7839, -19.0743, -2.1913, -0.9617, -0.2660, -0.1102, -0.0011\}$$

confirming that the LMI is fulfilled.

This approach does not necessary replace the direct verification of the fact that the Jacobian matrix (9) is stable. Such a verification implies the numerical calculation of the equilibrium point ($\widetilde{X} = 0.3206$, $\widetilde{Y_1} = 0.1006$, $\widetilde{Y_2} = 0.0084$) and also that of the eigenvalues of the Jacobian matrix

$$A = \begin{bmatrix} -13.0481 & -3.2059 & -1.6030 & 3.0000 & 0.1000 \\ -1.0062 & -17.2059 & 0 & 10.8000 & 0 \\ -0.0419 & 0 & -12.6030 & 0 & 6.0000 \\ 1.0062 & 3.2059 & 0 & -5.3000 & 0 \\ 0.0419 & 0 & 1.6030 & 0 & -5.3000 \end{bmatrix},$$

that is,

$$\Lambda_A = \{-20.4212, -12.2267, -13.8016, -2.8432, -4.1643\}.$$

All these eigenvalues are in the left half of the complex plane.

## 5. **Conclusions**

For those studying the micro RNA - messenger RNA dynamics our approach offers a sound numerical tool for checking the asymptotic stability of the system equilibrium in the positive ortant. It is worthwhile to mention that the proposed stability test is independent of the values of the equilibrium point, depending exclusively on the system coefficients.

Since Proposition 3.1 only provides a *sufficient condition* for verifying the asymptotic stability of the equilibrium point, a certain degree of conservatism is implicitly present in the numerical procedure; if the LMI (12) proves to be infeasible, this does not mean that the Jacobian matrix $A$ is unstable. Numerical experiments show that this conservatism becomes to be present for larger values of $\rho$.

Future work will be dedicated to the extension of the procedure to the situation $N > 1$ and to a better exploitation of the system's structure.

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