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SUPERCONVERGENCE OF THE STATIONARY INCOMPRESSIBLE MAGNETOHYDRODYNAMICS EQUATIONS

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A superconvergence result of the 2D stationary incompressible magnetohydrodynamics equations is constructed based on finite element method and L^2-projection technique, which, in fact, is a postprocessing procedure that establishes a new approximation based on a high order basic function on coarse mesh. Next, numerical experiments are presented to confirm correctness and effectiveness of theoretic analysis.

Keywords: magnetohydrodynamics, finite element method, superconvergence, L^2-projection technique.

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1. **Introduction**

Incompressible magnetohydrodynamics (MHD) problem is used to study the interaction between a viscous, incompressible, electrically conducting fluid and an external field. This strong nonlinear multiple variable coupling system is constructed by the Navier-stokes equations of hydrodynamics and the Maxwell's of electromagnetism via Lorentz force and Ohm's law. The model is very important and widely used in many areas, such as liquid metal cooling of nuclear reactors, process metallurgy and so on.

In this article, we will consider stationary incompressible MHD equations as follow [1]:

$$-R_e^{-1}\Delta\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p-S_c\mathrm{curl}\mathbf{B}\times\mathbf{B}=\mathbf{f}\quad\text{in }\Omega, \tag{1}$$

$$S_cR_m^{-1}\mathrm{curl}(\mathrm{curl}\mathbf{B})-S_c\mathrm{curl}(\mathbf{u}\times\mathbf{B})=\mathbf{g}\quad\text{in }\Omega, \tag{2}$$

$$\nabla\cdot\mathbf{u}=0\quad\text{in }\Omega, \tag{3}$$

$$\nabla\cdot\mathbf{B}=0\quad\text{in }\Omega, \tag{4}$$

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with the following boundary conditions

$$\mathbf{u} = 0 \quad \text{on} \quad \partial\Omega, \tag{5}$$

$$\mathbf{B} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega, \tag{6}$$

$$\mathbf{n} \times \text{curl}\mathbf{B} = 0 \quad \text{on} \quad \partial\Omega, \tag{7}$$

where $\Omega \subset R^2$, $\mathbf{n}$ is outward normal unit vector of $\partial\Omega$, R_e is hydrodynamic Reynolds number, R_m is magnetic Reynolds and S_c is coupling number. The MHD equations are characterized by three fields, and further discussion of them can be found in [2,3,4,5]. Here $\mathbf{u} = (u_1, u_2, 0)$, $\mathbf{B} = (B_1, B_2, 0)$ and p denote the velocity, magnetic and pressure field, respectively. And the known functions $\mathbf{f} = (f_1, f_2, 0)$ and $\mathbf{g} = (g_1, g_2, 0)$ are source terms.

In recent years, many studies have been devoted to the incompressible MHD equations by using finite element method (FEM). Gunzburger et al. [1] gave a detailed existence and uniqueness of the solutions of both the weak formulation and discrete Galerkin finite element schemes of MHD equations. In [6,7], the FEM for MHD problem was shown, which was based on weighted regularization analyzed by Hasler et al [8]. Besides, in order to violate the inf-sup condition, Gerbeau [3] and Salah et al. [4] developed and analyzed a stabilized finite element technique and Galerkin least-square method for MHD, respectively.

Superconvergence for finite element solutions has been an active research area in numerical analysis. The main objective in the superconvergence study is to improve the existing approximation accuracy by applying certain post-processing techniques. Several types of superconvergence in finite element methods have been studied in last two decades [9]. In this paper, we are concerned with the MHD problem and shall establish a superconvergence result for finite element approximations of the considered problem. We will apply a superconvergence technique called L^2-projection method proposed and analyzed by Wang and Ye [10]. The basic idea is to project the finite element solution to other finite element space on a coarser mesh. The difference of the two mesh sizes can be used to achieve a superconvergence result after post-processing procedure. For more details of this method, we refer the reader to the work of Chen and Wang [11], Heimsund et al. [12], Ye et al. [13-16], Liu and Yan [17], Li et al. [18,19] and Huang et al. [20,21].

The rest of the article is organized as follows. In the next section, an abstract functional setting of the 2D stationary MHD equations is given and then in Sect. 3 stability and convergence of standard FEM are recalled. In Sect. 4, a superconvergence result of the finite element solutions of the 2D stationary incompressible MHD problem based on L^2-projection method is constructed. In the final section, some numerical tests are provided to support the theoretical analysis.

2. Preliminaries

To get a weak form of (1)-(7), we employ the standard scalar Hilbert space $H^k(\Omega) = W^{k,2}(\Omega)$ for nonnegative integer k with norm $\|v\|_k = (\sum_{|\gamma|=0}^{k} \|D^\gamma v\|_0^2)^{\frac{1}{2}}$. For vector-value functions, we use the Hilbert space $\mathbf{H}^k(\Omega) = (H^k(\Omega))^2$ with norm $\|\mathbf{v}\|_k = (\sum_{i=1}^{2} \|v_i\|_k^2)^{\frac{1}{2}}$. Next, we introduce the following spaces [22,23]:

$$\mathbf{X} = \{\mathbf{w} \in \boldsymbol{H}^1(\Omega) : \mathbf{w}|_{\partial\Omega} = 0\},$$
$$\mathbf{W} = \{\mathbf{w} \in \boldsymbol{H}^1(\Omega) : \mathbf{w} \cdot \mathbf{n}|_{\partial\Omega} = 0\},$$
$$\mathbf{V} = \{\mathbf{w} \in \mathbf{X} : \mathrm{div}\mathbf{w} = 0 \text{ in } \Omega\},$$
$$\mathbf{V_n} = \{\mathbf{w} \in \mathbf{W} : \mathrm{div}\mathbf{w} = 0 \text{ in } \Omega\},$$

and

$$M = L_0^2(\Omega) = \left\{q \in L^2(\Omega) : \int_\Omega q d\mathbf{x} = 0\right\}.$$

We use the product space $\mathbf{W}_{0n} = \mathbf{X} \times \mathbf{W}$ equipped with the usual norm $\|(\mathbf{w}, \boldsymbol{\Phi})\|_1$ for all $(\mathbf{w}, \boldsymbol{\Phi}) \in \mathbf{W}_{0n}$ where $\|(\mathbf{w}, \boldsymbol{\Phi})\|_i = (\|\mathbf{w}\|_i^2 + \|\boldsymbol{\Phi}\|_i^2)^{\frac{1}{2}}$ (i=0,1,2). The space $\mathbf{H}^{-1}(\Omega)$ denotes the dual of $\mathbf{H}_0^1(\Omega)$ and $\|\cdot\|_{-1}$ represents the norm of dual space.

Now, we give the weak variational form of incompressible MHD system as follows: Find $((\mathbf{u}, \mathbf{B}), p) \in \mathbf{W}_{0n} \times M$ such that

$$\begin{aligned} &A_0((\mathbf{u}, \mathbf{B}), (\mathbf{v}, \boldsymbol{\Psi})) + A_1((\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}), (\mathbf{v}, \boldsymbol{\Psi})) - d((\mathbf{v}, \boldsymbol{\Psi}), p) + d((\mathbf{u}, \mathbf{B}), q) \\ &= (\mathbf{F}, (\mathbf{v}, \boldsymbol{\Psi})), \quad \forall((\mathbf{v}, \boldsymbol{\Psi}), q) \in \mathbf{W}_{0n} \times M, \end{aligned} \tag{8}$$

where

$$\begin{aligned}
&A_0((\mathbf{u}, \mathbf{B}), (\mathbf{v}, \boldsymbol{\Psi})) = a_0(\mathbf{u}, \mathbf{v}) + b_0(\mathbf{B}, \boldsymbol{\Psi}),\\
&a_0(\mathbf{w}, \mathbf{v}) = R_e^{-1}(\nabla\mathbf{w}, \nabla\mathbf{v}),\\
&b_0(\boldsymbol{\Phi}, \boldsymbol{\Psi}) = S_c R_m^{-1}(\nabla \times \boldsymbol{\Phi}, \nabla \times \boldsymbol{\Psi}) + S_c R_m^{-1}(\nabla \cdot \boldsymbol{\Phi}, \nabla \cdot \boldsymbol{\Psi}),\\
&A_1((\mathbf{u}, \mathbf{B}), (\mathbf{w}, \boldsymbol{\Phi}), (\mathbf{v}, \boldsymbol{\Psi})) = a_1(\mathbf{u}, \mathbf{w}, \mathbf{v}) - c(\boldsymbol{\Phi}, \mathbf{B}, \mathbf{v}) + c(\boldsymbol{\Psi}, \mathbf{B}, \mathbf{w}),\\
&a_1(\mathbf{u}, \mathbf{w}, \mathbf{v}) = (\mathbf{u} \cdot \nabla\mathbf{w} + \frac{1}{2}(\mathrm{div}\mathbf{u})\mathbf{w}, \mathbf{v}) = \frac{1}{2}(\mathbf{u} \cdot \nabla\mathbf{w}, \mathbf{v}) - \frac{1}{2}(\mathbf{u} \cdot \nabla\mathbf{v}, \mathbf{w}),\\
&c(\boldsymbol{\Phi}, \mathbf{B}, \mathbf{v}) = S_c(\mathrm{curl}\boldsymbol{\Phi} \times \mathbf{B}, \mathbf{v}),\\
&d((\mathbf{v}, \boldsymbol{\Psi}), q) = (\nabla \cdot \mathbf{v}, q),\\
&(\mathbf{F}, (\mathbf{v}, \boldsymbol{\Psi})) = (\mathbf{f}, \mathbf{v}) + (\mathbf{g}, \boldsymbol{\Psi}).
\end{aligned}$$

Further, we set

$$\|\mathbf{F}\|_{-1} = \sup_{(0,0)\neq(\mathbf{v},\boldsymbol{\Psi})\in\mathbf{W_{0n}}} \frac{(\mathbf{F}, (\mathbf{v}, \boldsymbol{\Psi}))}{\|(\mathbf{v}, \boldsymbol{\Psi})\|_1}.$$

The following properties for trilinear form $a_1(\cdot, \cdot, \cdot)$ are helpful to get the error estimates [23,24]:

$$a_1(\mathbf{u}, \mathbf{w}, \mathbf{v}) = -a_1(\mathbf{u}, \mathbf{v}, \mathbf{w}), \quad \forall\mathbf{u}, \mathbf{w}, \mathbf{v} \in \mathbf{X}, \tag{9}$$

$$|a_1(\mathbf{u}, \mathbf{w}, \mathbf{v})| \leq C_0^2\|\nabla\mathbf{u}\|_0\|\nabla\mathbf{w}\|_0\|\nabla\mathbf{v}\|_0, \quad \forall\mathbf{u}, \mathbf{w}, \mathbf{v} \in \mathbf{X}, \tag{10}$$

where C_0 depends only on Ω. Obviously, $A_1(\cdot, \cdot, \cdot)$ satisfies

$$A_1((\mathbf{u}, \mathbf{B}), (\mathbf{w}, \boldsymbol{\Phi}), (\mathbf{w}, \boldsymbol{\Phi})) = 0, \quad \forall(\mathbf{u}, \mathbf{B}), (\mathbf{w}, \boldsymbol{\Phi}) \in \mathbf{W}_{0n}. \tag{11}$$

Next, we recall the coercive and continuous properties of $A_0(\cdot,\cdot)$ and the continuous property of $A_1(\cdot,\cdot,\cdot)$ [1] as follows based on the following two inequalities

$$\|\mathrm{curl}\mathbf{v}\|_0 \leq \sqrt{2}\|\nabla\mathbf{v}\|_0,$$

and

$$\|\nabla\cdot\mathbf{v}\|_0 \leq \|\nabla\mathbf{v}\|_0,$$

which are shown in [23,25]. For all $(\mathbf{u},\mathbf{B}),(\mathbf{w},\boldsymbol{\Phi}),(\mathbf{v},\boldsymbol{\Psi}) \in \mathbf{W}_{0n}$, there hold

$$A_0((\mathbf{w},\boldsymbol{\Phi})),(\mathbf{w},\boldsymbol{\Phi}))) \geq \nu_1\|(\mathbf{w},\boldsymbol{\Phi})\|_1^2, \tag{12}$$

$$A_0((\mathbf{w},\boldsymbol{\Phi})),(\mathbf{v},\boldsymbol{\Psi}))) \leq \nu_2\|(\mathbf{w},\boldsymbol{\Phi})\|_1\|(\mathbf{v},\boldsymbol{\Psi})\|_1, \tag{13}$$

$$A_1((\mathbf{u},\mathbf{B}),(\mathbf{w},\boldsymbol{\Phi}),(\mathbf{v},\boldsymbol{\Psi}) \leq N\|(\mathbf{u},\mathbf{B})\|_1\|(\mathbf{w},\boldsymbol{\Phi})\|_1\|(\mathbf{v},\boldsymbol{\Psi})\|_1, \tag{14}$$

where

$$\nu_1 = \min\{R_e^{-1}, C_1S_cR_m^{-1}\}, \quad \nu_2 = \max\{R_e^{-1}, 3S_cR_m^{-1}\}, \quad N = \sqrt{2}C_0^2\max\{1, \sqrt{2}S_c\},$$

and C_1 (depending only on Ω) is an embedding constant of $\boldsymbol{W} \hookrightarrow \mathbf{H}^1(\Omega)$, i.e. ,

$$\|\nabla\times\boldsymbol{\Psi}\|_0^2 + \|\nabla\cdot\boldsymbol{\Psi}\|_0^2 \geq C_1\|\boldsymbol{\Psi}\|_1^2, \quad \forall\boldsymbol{\Psi}\in\mathbf{W}. \tag{15}$$

Final, we recall the existence and uniqueness results in [23] as follows.

Theorem 2.1 If R_e, S_c and R_m satisfy the uniqueness condition

$$0 < \frac{N\|\mathbf{F}\|_{-1}}{\nu_1^2} < 1, \tag{16}$$

there exists a unique solution pair $((\mathbf{u},\mathbf{B}),p) \in \mathbf{W}_{0n}\times M$ in problem (8) and satisfies

$$\nu_1\|(\mathbf{u},\mathbf{B})\|_1 \leq \|\mathbf{F}\|_{-1}. \tag{17}$$

At the end of the section, we consider the following linear problem: find $((\mathbf{w},\boldsymbol{\Phi}),s) \in \mathbf{W}_{0n}\times M$ such that for all $((\mathbf{v},\boldsymbol{\Psi}),q) \in \mathbf{W}_{0n}\times M$,

$$\begin{cases} A_0((\mathbf{v},\boldsymbol{\Psi}),(\mathbf{w},\boldsymbol{\Phi})) + A_1((\mathbf{u},\mathbf{B}),(\mathbf{v},\boldsymbol{\Psi}),(\mathbf{w},\boldsymbol{\Phi})) + A_1((\mathbf{v},\boldsymbol{\Psi}),(\mathbf{u},\mathbf{B}),(\mathbf{w},\boldsymbol{\Phi})) - d((\mathbf{v},\boldsymbol{\Psi}),p) \\ = (\mathbf{F},(\mathbf{v},\boldsymbol{\Psi})), \\ d((\mathbf{w},\boldsymbol{\Phi}),q) = (\gamma,q). \end{cases} \tag{18}$$

Assume the domain Ω is convex polygonal. Moreover, supposing $\mathbf{f},\mathbf{g} \in \mathbf{L}^2(\Omega)$ and $\gamma \in L^2(\Omega)$, the solution $((\mathbf{w},\boldsymbol{\Phi}),s)$ of the problem (18) satisfies the following regularity [23]

$$\|(\mathbf{w},\boldsymbol{\Phi})\|_2 + \|s\|_1 \leq C(\|\mathbf{f}\|_0 + \|\mathbf{g}\|_0 + \|\gamma\|_1). \tag{19}$$

Throughout the paper, the letter C represents a general positive constant meaning for different values at different places, which has nothing to do with mesh size.

3. **Finite element approximation**

Now, let T_h be a quasi-uniform and regular partition of Ω into triangles with the mesh parameter h ($h \to 0$). Based on the T_h, the finite element spaces $(\mathbf{X_h}, M_h, \mathbf{W_h})$ are constructed. Then, we introduce the following assumption on $(\mathbf{X_h}, M_h, \mathbf{W_h})$ in [1, 2].

Assumption A$_1$ Let $\mathbf{W}_{0n}^h = \mathbf{X_h} \times \mathbf{W_h}$. There exists a constant β (only dependent on Ω), such that

$$\sup_{(\mathbf{v},\mathbf{\Psi})\in \mathbf{W}_{0n}^h} \frac{d((\mathbf{v},\mathbf{\Psi}),q)}{\|(\mathbf{v},\mathbf{\Psi})\|_1} \geq \beta \|q\|_0, \forall q \in M_h.$$

Moveover, we employ the following Mini-element used traditionally for the Navier-Stokes equations to approximate velocity and pressure which satisfies the above assumption, and any appropriate subspace of $\mathbf{H}^1(\Omega)$ to approximate magnetic filed, i.e.,

$$\mathbf{X}_h = (P_1^b)^2 \cap \mathbf{X}, \quad M_h = \left\{ q_h \in C^0(\Omega) : v_h|_K \in P_1(K), \forall K \subset T_h \right\},$$

where

$$P_1^b = \left\{ v_h \in C^0(\Omega) : v_h|_K \in P_1(K) \oplus \text{span}\{\hat{b}\}, \forall K \subset T_h \right\},$$

$\hat{b}$ is a bubble function, and $P_1(K)$ denotes the space of polynomials of degree less than or equal to 1 on element K. For convenience, we use $\mathbf{W}_h = (P_1^b)^2 \cap \mathbf{W}$ for approximation of magnetic field. So, the Galerkin finite element scheme of (8) is to seek $((\mathbf{u}_h, \mathbf{B}_h), p_h) \in \mathbf{W}_{0n}^h \times M_h$ such that

$$\begin{aligned} &A_0((\mathbf{u}_h, \mathbf{B}_h), (\mathbf{v}, \mathbf{\Psi})) + A_1((\mathbf{u}_h, \mathbf{B}_h), (\mathbf{u}_h, \mathbf{B}_h), (\mathbf{v}, \mathbf{\Psi})) - d((\mathbf{v}, \mathbf{\Psi}), p_h) + d((\mathbf{u}_h, \mathbf{B}_h), q) \\ &= (\mathbf{F}, (\mathbf{v}, \mathbf{\Psi})), \quad \forall ((\mathbf{v}, \mathbf{\Psi}), q) \in \mathbf{W}_{0n}^h \times M_h. \end{aligned} \tag{20}$$

Besides, we need to recall the following important theorem in [23] which is necessary for superconvergence result that we want to establish.

Theorem 3.1 Under the assumptions of Theorem 2.1 and Assumption A$_1$, the finite element scheme (20) has a unique solution pair $((\mathbf{u}_h, \mathbf{B}_h), p_h) \in \mathbf{W}_{0n}^h \times M_h$ which satisfies

$$\nu_1 \|(\mathbf{u}_h, \mathbf{B}_h)\|_1 \leq \|\mathbf{F}\|_{-1}, \tag{21}$$

and the error estimate

$$\|(\mathbf{u} - \mathbf{u}_h, \mathbf{B} - \mathbf{B}_h)\|_0 + h\Big(\nu_1 \|(\mathbf{u} - \mathbf{u}_h, \mathbf{B} - \mathbf{B}_h)\|_1 + \|p - p_h\|_0\Big) \leq Ch^2 \|\mathbf{F}\|_0. \tag{22}$$

4. **Superconvergence result**

From the above section, the finite element partition T_h was used to produce the finite element approximation $((\mathbf{u}_h, \mathbf{B}_h), p_h)$ in (20), and now in order to get a superconvergence result, we introduce another three finite element partitions T_{ρ_i}

with mesh sizes ρ_i, where $h \ll \rho_i$ $(i = 1, 2, 3)$. Assume that these partitions are quasi-uniform and regular. Let ρ_i and h have the following relationship:

$$\rho_i = h^{\sigma_i},$$

where $\sigma_i \in (0, 1)$. The parameter σ_i will have a great effect on achieving a superconvergence for the finite element approximation $((\mathbf{u}_h, \mathbf{B}_h), p_h)$. Then we employ some finite element spaces $\boldsymbol{W}_{on}^{(\rho_1,\rho_2)}$ and M_{ρ_3}, which consist of piecewise polynomials of degree δ_1, δ_2 and δ_3, associated with the partitions T_{ρ_1}, T_{ρ_2} and T_{ρ_3}, respectively.

Next, we define $\mathbf{Q}_{(\rho_1,\rho_2)}$ and H_{ρ_3} to be L^2-projectors from $\boldsymbol{L}^2(\Omega)^2$ and $L^2(\Omega)$ onto the finite element spaces $\mathbf{W}_{on}^{(\rho_1,\rho_2)}$ and M_{ρ_3}, respectively. Note that these L^2-projectors are the self-adjoint operators [26], and they have the following properties [27]:

$$\|\mathbf{Q}_{(\rho_1,\rho_2)}\boldsymbol{\phi}\|_0 \le C\|\boldsymbol{\phi}\|_0, \quad \forall \boldsymbol{\phi} \in \boldsymbol{L}^2(\Omega)^2, \qquad \|\mathrm{H}_{\rho_3}\phi\|_0 \le C\|\phi\|_0, \quad \forall \phi \in L^2(\Omega).$$

In fact, the post-processing of the finite element approximation $((\mathbf{u}_h, \mathbf{B}_h), p_h)$ is simply given by their L^2-projections. Hence, we arrive at the superconvergence result as follows

$$\textit{superconvergence}\,((\mathbf{u}_h, \mathbf{B}_h), p_h) := \textit{post-processed}\,((\mathbf{u}_h, \mathbf{B}_h), p_h) = (\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u}_h, \mathbf{B}_h), \mathrm{H}_{\rho_3} p_h).$$

Moreover, subtracting (8) from (20), we obtain the error equation

$$\begin{aligned} &A_0((\mathbf{u}-\mathbf{u}_h, \mathbf{B}-\mathbf{B}_h), (\mathbf{v}, \mathbf{\Psi})) + A_1((\mathbf{u}-\mathbf{u}_h, \mathbf{B}-\mathbf{B}_h), (\mathbf{u}, \mathbf{B}), (\mathbf{v}, \mathbf{\Psi})) \\ &+ A_1((\mathbf{u}_h, \mathbf{B}_h), (\mathbf{u}-\mathbf{u}_h, \mathbf{B}-\mathbf{B}_h), (\mathbf{v}, \mathbf{\Psi})) - d((\mathbf{v}, \mathbf{\Psi}), p - p_h) \\ &+ d((\mathbf{u}-\mathbf{u}_h, \mathbf{B}-\mathbf{B}_h), q) = 0, \quad \forall ((\mathbf{v}, \mathbf{\Psi}), q) \in \mathbf{W}_{0n}^h \times M_h, \end{aligned} \tag{23}$$

and the above error equation will play an important role in proof of following lemmas. Now we provide two important lemmas, which will be useful for establishing superconvergence result.

Lemma 4.1 Assume (19) holds and $\mathbf{W}_{on}^{(\rho_1,\rho_2)} \subset \boldsymbol{L}^2(\Omega)^2$. Under the assumption of Theorem 3.1, there exists a constant C such that

$$\|\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u}-\mathbf{u}_h, \mathbf{B}-\mathbf{B}_h)\|_0 \le Ch^2\|\mathbf{F}\|_0.$$

Proof. Consider the following problem: Find $((\mathbf{w}, \mathbf{\Phi}), s) \in \mathbf{W}_{0n} \times M$ such that for all $((\mathbf{v}, \mathbf{\Psi}), q) \in \mathbf{W}_{0n} \times M$,

$$\begin{cases} A_0((\mathbf{v}, \mathbf{\Psi}), (\mathbf{w}, \mathbf{\Phi})) + A_1((\mathbf{u}, \mathbf{B}), (\mathbf{v}, \mathbf{\Psi}), (\mathbf{w}, \mathbf{\Phi})) + A_1((\mathbf{v}, \mathbf{\Psi}), (\mathbf{u}, \mathbf{B}), (\mathbf{w}, \mathbf{\Phi})) - d((\mathbf{v}, \mathbf{\Psi})), s) \\ \quad = (\mathbf{Q}_{(\rho_1,\rho_2)}\phi, (\mathbf{v}, \mathbf{\Psi})), \\ d((\mathbf{w}, \mathbf{\Phi}), q) = 0, \end{cases} \tag{24}$$

for any $\phi \in (C_0^\infty(\Omega))^2$. Subtracting (23) with $((\mathbf{v},\boldsymbol{\Psi}),q) = ((\mathbf{w}_h,\boldsymbol{\Phi}_h),s_h)$ from (24) with $((\mathbf{v},\boldsymbol{\Psi}),q) = ((\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h),p-p_h)$, we get

$$
\begin{aligned}
&(\mathbf{Q}_{(\rho_1,\rho_2)}\phi,(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h))\\
&= A_0((\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h),(\mathbf{w}-\mathbf{w}_h,\boldsymbol{\Phi}-\boldsymbol{\Phi}_h)) + A_1((\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h),(\mathbf{u},\mathbf{B}),(\mathbf{w}-\mathbf{w}_h,\boldsymbol{\Phi}-\boldsymbol{\Phi}_h))\\
&+ A_1((\mathbf{u},\mathbf{B}),(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h),(\mathbf{w}-\mathbf{w}_h,\boldsymbol{\Phi}-\boldsymbol{\Phi}_h))\\
&- A_1((\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h),(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h),(\mathbf{w}-\mathbf{w}_h,\boldsymbol{\Phi}-\boldsymbol{\Phi}_h))\\
&+ A_1((\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h),(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h),(\mathbf{w},\boldsymbol{\Phi}))\\
&- d((\mathbf{w}-\mathbf{w}_h,\boldsymbol{\Phi}-\boldsymbol{\Phi}_h),p-p_h) - d((\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h),s-s_h),
\end{aligned}
\tag{25}
$$

for all $((\mathbf{w}_h,\boldsymbol{\Phi}_h),s_h) \in \mathbf{W}_{0n}^h \times M_h$. Using (13), (14) and (22), we arrive at

$$
\begin{aligned}
&(\mathbf{Q}_{(\rho_1,\rho_2)}\phi,(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h))\\
&\le \nu_2\|(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h)\|_1\|(\mathbf{w}-\mathbf{w}_h,\boldsymbol{\Phi}-\boldsymbol{\Phi}_h)\|_1\\
&+ 2N\|(\mathbf{u},\mathbf{B})\|_1\|(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h)\|_1\|(\mathbf{w}-\mathbf{w}_h,\boldsymbol{\Phi}-\boldsymbol{\Phi}_h)\|_1\\
&+ N\|(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h)\|_1{}^2(\|(\mathbf{w}-\mathbf{w}_h,\boldsymbol{\Phi}-\boldsymbol{\Phi}_h)\|_1 + \|(\mathbf{w},\boldsymbol{\Phi})\|_1) + \|(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h)\|_1\|s-s_h\|_0\\
&+ \|(\mathbf{w}-\mathbf{w}_h,\boldsymbol{\Phi}-\boldsymbol{\Phi}_h)\|_1\|p-p_h\|_0\\
&\le C\Big(h\Big(\|(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h)\|_1 + \|p-p_h\|_0\Big) + \|(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h)\|_1^2\Big)\Big(\|(\mathbf{w},\boldsymbol{\Phi})\|_2 + \|s\|_1\Big)\\
&\le Ch^2\|\mathbf{F}\|_0\Big(\|(\mathbf{w},\boldsymbol{\Phi})\|_2 + \|s\|_1\Big).
\end{aligned}
$$

Moreover, applying the regularity (19) and the property of L^2-projection yields

$$
\|(\mathbf{w},\boldsymbol{\Phi})\|_2 + \|s\|_1 \le C\|\mathbf{Q}_{(\rho_1,\rho_2)}\phi\|_0 \le C\|\phi\|_0, \tag{26}
$$

Then combining with the aforementioned bound, we lead to

$$
\begin{aligned}
\|\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h)\|_0 &= \sup_{\phi\in(C_0^\infty(\Omega))^2,\phi\neq 0} \frac{|\,(\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h),\phi)\,|}{\|\phi\|_0}\\
&= \sup_{\phi\in(C_0^\infty(\Omega))^2,\phi\neq 0} \frac{|\,((\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h),\mathbf{Q}_{(\rho_1,\rho_2)}\phi)\,|}{\|\phi\|_0} \le Ch^2\|\mathbf{F}\|_0,
\end{aligned}
$$

where the second equation is deduced by property of the self-adjoint operator $\mathbf{Q}_{(\rho_1,\rho_2)}$ (see Theorem 12.14, [26]).

The proof of the lemma completes.

Lemma 4.2 Assuming (19), $\mathrm{H}_{\rho_3} \in H^1(\Omega)$ and Theorem 3.1 hold, there exists a constant C such that

$$
\|\mathrm{H}_{\rho_3}\mathrm{p} - \mathrm{H}_{\rho_3}\mathrm{p}_h\|_0 \le Ch^{2-\sigma_3}\|\mathbf{F}\|_0.
$$

Proof. By the definitions of $\|\cdot\|_0$ and H_{ρ_3}, we have

$$\|\mathrm{H}_{\rho_3}\mathrm{p}-\mathrm{H}_{\rho_3}\mathrm{p}_h\|_0=\sup_{\phi\in C_0^\infty(\Omega),\phi\neq 0}\frac{|\,(\mathrm{H}_{\rho_3}\mathrm{p}-\mathrm{H}_{\rho_3}\mathrm{p}_h,\phi)\,|}{\|\phi\|_0}=\sup_{\phi\in C_0^\infty(\Omega),\phi\neq 0}\frac{|\,(\mathrm{p}-\mathrm{p}_h,\mathrm{H}_{\rho_3}\phi)\,|}{\|\phi\|_0}.$$

As the proof of Lemma 4.1, here the second equation is deduced by property of the self-adjoint operator H_{ρ_3}.

Next, consider the following dual linear problem: seeking $((\boldsymbol{\zeta},\boldsymbol{\beta}),\theta)\in\mathbf{W}_{0n}\times M$ such that for all $((\mathbf{v},\boldsymbol{\Psi}),q)\in\mathbf{W}_{0n}\times M$,

$$\begin{cases}A_0((\mathbf{v},\boldsymbol{\Psi}),(\boldsymbol{\zeta},\boldsymbol{\beta}))+A_1((\mathbf{u},\mathbf{B}),(\mathbf{v},\boldsymbol{\Psi}),(\boldsymbol{\zeta},\boldsymbol{\beta}))+A_1((\mathbf{v},\boldsymbol{\Psi}),(\mathbf{u},\mathbf{B}),(\boldsymbol{\zeta},\boldsymbol{\beta}))-d((\mathbf{v},\boldsymbol{\Psi}),\theta)=0,\\ d((\boldsymbol{\zeta},\boldsymbol{\beta}),q)=(\mathrm{H}_{\rho_3}\phi,q).\end{cases}\tag{27}$$

In (27) and (23), choose $((\mathbf{v},\boldsymbol{\Psi}),q)=((\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h),p-p_h)$ and $((\mathbf{v},\boldsymbol{\Psi}),q)=((\zeta_h,\boldsymbol{\beta}_h),\theta_h)$, respectively, and subtracting (23) from (27), we get

$$\begin{aligned}(\mathrm{H}_{\rho_3}\phi,p-p_h)=\ &A_0((\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h),(\boldsymbol{\zeta}-\zeta_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h))\\ &+A_1((\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h),(\mathbf{u},\mathbf{B}),(\boldsymbol{\zeta}-\zeta_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h))\\ &+A_1((\mathbf{u},\mathbf{B}),(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h),(\boldsymbol{\zeta}-\zeta_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h))\\ &-A_1((\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h),(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h),(\boldsymbol{\zeta}-\zeta_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h))\\ &+A_1((\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h),(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h),(\boldsymbol{\zeta},\boldsymbol{\beta}))\\ &-d((\boldsymbol{\zeta}-\zeta_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h),p-p_h)-d((\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h),\theta-\theta_h),\end{aligned}$$

for all $((\zeta_h,\boldsymbol{\beta}_h),\theta_h)\in\mathbf{W}_{0n}^h\times M_h$. Similarly, using (13), (14), (19) and (22), we arrive at

$$(\mathrm{H}_{\rho_3}\phi,p-p_h)\leq Ch^2\|\mathbf{F}\|_0\|\mathrm{H}_{\rho_3}\phi\|_1.\tag{28}$$

Further, by using the above property of projection H_{ρ_3} and the inverse inequality [28]:

$$\|\mathrm{H}_{\rho_3}\phi\|_1\leq C\rho_3^{-1}\|\mathrm{H}_{\rho_3}\phi\|_0,\quad\forall\phi\in L^2(\Omega),$$

we arrive at

$$\begin{aligned}\|\mathrm{H}_{\rho_3}p-\mathrm{H}_{\rho_3}p_h\|_0&=\sup_{\phi\in C_0^\infty(\Omega),\phi\neq 0}\frac{|\,(\mathrm{p}-\mathrm{p}_h,\mathrm{H}_{\rho_3}\phi)\,|}{\|\phi\|_0}\leq\sup_{\phi\in C_0^\infty(\Omega),\phi\neq 0}\frac{Ch^2\|\mathbf{F}\|_0\|\mathrm{H}_{\rho_3}\phi\|_1}{\|\phi\|_0}\\ &\leq Ch^2\rho_3^{-1}\|\mathbf{F}\|_0\leq Ch^{2-\sigma_3}\|\mathbf{F}\|_0.\end{aligned}$$

Then the proof of the lemma completes.

Now, we are in a position to estimate $(\mathbf{u},\mathbf{B})-\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u}_h,\mathbf{B}_h)$ and $p-\mathrm{H}_{\rho_3}p_h$.

Theorem 4.1 Under the above mentioned assumptions of lemmas 4.1 and 4.2, if ρ_i,σ_i $(i=1,2)$ and h satisfy $\rho_i=h^{\sigma_i}$ with $\sigma_i=\frac{2}{\delta_i+1}$, then

$$\|(\mathbf{u},\mathbf{B})-\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u}_h,\mathbf{B}_h)\|_0\leq Ch^2(\|(\mathbf{u},\mathbf{B})\|_{\delta_i+1}+\|\mathbf{F}\|_0),$$

and

$$\|\nabla_{(\rho_1,\rho_2)}((\mathbf{u},\mathbf{B})-\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u}_h,\mathbf{B}_h))\|_0 \le Ch^{\frac{2\delta_i}{1+\delta_i}}(\|(\mathbf{u},\mathbf{B})\|_{\delta_i+1}+\|\mathbf{F}\|_0),$$

where $\nabla_{(\rho_1,\rho_2)}$ is defined element-wise over the partition T_{ρ_1} and T_{ρ_2}. Besides, if ρ_3, σ_3 and h satisfy $\rho_3 = h^{\sigma_3}$ with $\sigma_3 = \frac{2}{\delta_3+2}$, then

$$\|p - \mathrm{H}_{\rho_3} p_h\|_0 \le Ch^{\frac{2(\delta_3+1)}{\delta_3+2}}(\|p\|_{\delta_3+1}+\|\mathbf{F}\|_0).$$

Proof. By the definitions of $\mathbf{Q}_{(\rho_1,\rho_2)}$ and H_{ρ_3}, we obtain

$$\begin{aligned}\|(\mathbf{u},\mathbf{B})-\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u},\mathbf{B})\|_0 + \rho_i\|\nabla_{(\rho_1,\rho_2)}((\mathbf{u},\mathbf{B})-\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u},\mathbf{B}))\|_0 &\le C\rho_i^{\delta_i+1}\|(\mathbf{u},\mathbf{B})\|_{\delta_i+1}\\ &= Ch^{\sigma_i(\delta_i+1)}\|(\mathbf{u},\mathbf{B})\|_{\delta_i+1},\\ \|p-\mathrm{H}_{\rho_3}p_h\|_0 \le C\rho_3^{\delta_3+1}\|p\|_{\delta_3+1} &= Ch^{\sigma_3(\delta_3+1)}\|p\|_{\delta_3+1}.\end{aligned}$$

Moreover, combining the Minkowski inequality, the lemmas 4.1 and 4.2 yields

$$\begin{aligned}\|(\mathbf{u},\mathbf{B})-\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u}_h,\mathbf{B}_h)\|_0 &\le \|(\mathbf{u},\mathbf{B})-\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u},\mathbf{B})\|_0 + \|\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u},\mathbf{B})-\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u}_h,\mathbf{B}_h)\|_0\\ &\le C(h^{\sigma_i(\delta_i+1)}\|(\mathbf{u},\mathbf{B})\|_{\delta_i+1}+h^2\|\mathbf{F}\|_0),\end{aligned} \tag{29}$$

$$\|p-\mathrm{H}_{\rho_3}p_h\|_0 \le C(h^{\sigma_3(\delta_3+1)}\|p\|_{\delta_3+1}+h^{2-\sigma_3}\|\mathbf{F}\|_0). \tag{30}$$

Then the error estimate (29) and (30) can be optimized by choosing $\sigma_i = \frac{2}{\delta_i+1}$ and σ_3 such that $\sigma_3(\delta_3+1) = 2-\sigma_3$ (that is $\sigma_3 = \frac{2}{\delta_3+2}$), respectively. Until now, we have got the L^2-estimate of $(\mathbf{u},\mathbf{B})-\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u}_h,\mathbf{B}_h)$ and $p-\mathrm{H}_{\rho_3}p_h$.

Next, by using the inverse inequality and Lemma 4.1, we have

$$\begin{aligned}&\|\nabla_{(\rho_1,\rho_2)}((\mathbf{u},\mathbf{B})-\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u}_h,\mathbf{B}_h))\|_0\\ &\le \|\nabla_{(\rho_1,\rho_2)}((\mathbf{u},\mathbf{B})-\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u},\mathbf{B}))\|_0 + \|\nabla_{(\rho_1,\rho_2)}(\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u},\mathbf{B})-\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u}_h,\mathbf{B}_h))\|_0\\ &\le C(h^{\sigma_i\delta_i}\|(\mathbf{u},\mathbf{B})\|_{\delta_i+1}+h^{2-\sigma_i}\|\mathbf{F}\|_0).\end{aligned}$$

Then optimizing the above-mentioned estimate by choosing such a σ_i that $\sigma_i\delta_i = 2-\sigma_i$, we can get the following estimate:

$$\|\nabla_{(\rho_1,\rho_2)}((\mathbf{u},\mathbf{B})-\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u}_h,\mathbf{B}_h))\|_0 \le Ch^{\frac{2\delta_i}{1+\delta_i}}(\|(\mathbf{u},\mathbf{B})\|_{\delta_i+1}+\|\mathbf{F}\|_0). \tag{31}$$

Combining (29)-(31), we complete the proof.

5. **Assessment of superconvergence**

In this section, we will give computational results for the stationary incompressible MHD equations. Our primary aim is to obtain superconvergence result. For current investigation, an uniform mesh is adopted; that is, the mesh consists of triangular elements that are obtained by dividing Ω into subsquares of equal size and then drawing the diagonal in each subsquare.

Let the solution domain $\Omega = [0,1] \times [0,1]$. The right-hand sides of MHD equations are determined by the exact solutions (for the velocity $\mathbf{u} = (u_1, u_2)$, the magnetic $\mathbf{B} = (B_1, B_2)$ and the pressure p) given by

$$u_1(x,y) = \pi \sin(\pi y)\cos(\pi y)\sin^2(\pi x), \qquad u_2(x,y) = -\pi \sin(\pi x)\cos(\pi x)\sin^2(\pi y),$$
$$B_1(x,y) = \sin(\pi x)\cos(\pi y), \qquad B_2(x,y) = -\sin(\pi y)\cos(\pi x),$$
$$p(x,y) = \cos(\pi x)\cos(\pi y).$$

This example is aimed to confirm the predicted superconvergence rate of the finite element approximation for the stationary MHD equations by the L^2-projection method.

We take $S_c = 1, R_e = 1$ and $R_m = 1$. From Theorem 3.1, we know that the finite element solution pair $((\mathbf{u}_h, \mathbf{B}_h), p_h)$ has the optimal error estimate. Moreover, in order to achieve superconvergence for the numerical solution, the L^2-projection method is applied. The key of this technique is to project one finite element space onto other finite element space based on a high order of polynomial on coarse mesh. In Table 1, we present a theoretical superconvergence result of the finite element solution by Theorem 4.1, where $\delta_1 = \delta_2 = 2, \delta_3 = 1$.

Table 1 Superconvergence result by projecting $(P_1b)^2 - P_1$ to $(P_2)^2 - P_1$.

FEM solutions	mesh			$\mathbf{u}_{L^2} - rate$	$\mathbf{u}_{H^1} - rate$	$\mathbf{B}_{L^2} - rate$	$\mathbf{B}_{H^1} - rate$	$P_{L^2} - rate$
$((\mathbf{u}_h, \mathbf{B}_h), p_h)$	h	h	h	2	1	2	1	1
$(\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u}_h, \mathbf{B}_h), \mathrm{H}_{\rho_3} p_h)$	$h^{\frac{2}{3}}$	$h^{\frac{2}{3}}$	$h^{\frac{2}{3}}$	2	$\frac{4}{3}$	2	$\frac{4}{3}$	$\frac{4}{3}$

On the one hand, we test the convergence rate of the finite element solution for the stationary MHD equations. Table 2 shows the convergence rates with different mesh sizes by $(P_1b)^2 - P_1$ element. The numerical convergence rate for the pressure in the L^2-norm seems better than the theoretical one.

Table 2 The convergence rate of $(P_1b)^2 - P_1$.

$\frac{1}{h}$	$\frac{\|\mathbf{u}-\mathbf{u}_h\|_0}{\|\mathbf{u}\|_0}$	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_0}{\|\nabla\mathbf{u}\|_0}$	$\frac{\|\mathbf{B}-\mathbf{B}_h\|_0}{\|\mathbf{B}\|_0}$	$\frac{\|\nabla(\mathbf{B}-\mathbf{B}_h)\|_0}{\|\nabla\mathbf{B}\|_0}$	$\frac{\|p-p_h\|_0}{\|p\|_0}$
10	6. 77e-2	2. 42e-1	2. 51e-2	1. 48e-1	1. 37e-0
20	1. 71e-2	1. 21e-1	6. 36e-3	7. 44e-2	4. 38e-1
30	7. 61e-3	8. 08e-2	2. 83e-3	4. 96e-2	2. 31e-1
40	4. 28e-3	6. 06e-2	1. 60e-3	3. 72e-2	1. 48e-1
50	2. 74e-3	4. 84e-2	1. 02e-3	2. 98e-2	1. 05e-1
Rate	2.00	1.00	2.00	1.00	1.53

On the other hand, we check the superconvergence result of the FEM developed for the considered problem. By applying the L^2-projection method, the solution $(\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u}_h, \mathbf{B}_h), \mathrm{H}_{\rho_3} p_h)$ is given by follows: Find $(\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u}_h, \mathbf{B}_h), \mathrm{H}_{\rho_3} p_h) \in \boldsymbol{W}_{on}^{(\rho_1,\rho_2)} \times \mathrm{M}_{\rho_3}$ for all $((\mathbf{v}, \boldsymbol{\Psi}), q) \in \boldsymbol{W}_{on}^{(\rho_1,\rho_2)} \times \mathrm{M}_{\rho_3}$ such that

$$(\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u}_h, \mathbf{B}_h), (\mathbf{v}, \boldsymbol{\Psi})) = ((\mathbf{u}_h, \mathbf{B}_h), (\mathbf{v}, \boldsymbol{\Psi})), \quad (\mathrm{H}_{\rho_3} p_h, q) = (p_h, q),$$

where $((\mathbf{u}_h, \mathbf{B}_h), p_h)$ is the solution obtained by the FEM. The error estimates for $(\mathbf{Q}_{(\rho_1,\rho_2)}(\mathbf{u}_h, \mathbf{B}_h), \mathrm{H}_{\rho_3}p_h)$ are given in Theorem 4.1. In Table 3, five values of h are chosen, and then the corresponding ρ_i $(i = 1, 2, 3)$ is obtained by using $\rho_i = h^{\sigma_i}$, $\sigma_i = \frac{2}{3}$. Numerical result by projecting $(P_1b)^2 - P_1$ to $(P_2)^2 - P_1$ is listed and it confirms the superconvergence result provided in Theorem 4.1.

Table 3 The superconvergence result by projecting $(P_1b)^2 - P_1$ to $(P_2)^2 - P_1$.

$\frac{1}{\mathrm{h}}$	$\frac{1}{\rho_i}$	$\frac{\|\mathbf{u}-\mathbf{Q}_{\rho_1}\mathbf{u}_h\|_0}{\|\mathbf{u}\|_0}$	$\frac{\|\nabla_{\rho_1}(\mathbf{u}-\mathbf{Q}_{\rho_1}\mathbf{u}_h)\|_0}{\|\nabla\mathbf{u}\|_0}$	$\frac{\|\mathbf{B}-\mathbf{Q}_{\rho_2}\mathbf{B}_h\|_0}{\|\mathbf{B}\|_0}$	$\frac{\|\nabla_{\rho_2}(\mathbf{B}-\mathbf{Q}_{\rho_2}\mathbf{B}_h)\|_0}{\|\nabla\mathbf{B}\|_0}$	$\frac{\|p-H_{\rho_3}p_h\|_0}{\|p\|_0}$
10	5	7. 15e-2	1. 71e-1	2. 53e-2	6. 57e-2	7. 06e-1
20	7	1. 71e-2	6. 33e-2	6. 29e-3	2. 44e-2	2. 08e-1
30	10	7. 78e-3	3. 82e-2	2. 79e-3	1. 40e-2	1. 10e-1
40	12	4. 37e-3	2. 48e-2	1. 56e-3	8. 41e-3	7. 15e-2
50	14	2. 77e-3	1. 80e-2	9. 98e-4	6. 05e-3	5. 22e-2
Rate		2.06	1.42	2.01	1.48	1.41

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