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AUXILIARY PRINCIPLE TECHNIQUE FOR STRONGLY MIXED VARIATIONAL-LIKE INEQUALITIES

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In this paper, we introduce a new class of variational-like inequalities, which is called strongly mixed variational-like inequality. It is shown that the optimality conditions for the sum of differentiable preinvex functions and nondifferentiable strongly preinvex functions can be characterized by strongly mixed variational-like inequalities. We use the auxiliary principle technique to study the existence of a solution of the strongly mixed variational-like inequalities. Some iterative methods for solving strongly mixed variational-like inequalities are suggested. Convergence analysis of these proposed methods is considered under mild conditions. Some special cases are also discussed which can be obtained from our results.

Keywords: Strongly mixed variational inequality, Iterative method, Convergence

MSC 2000: 26D15, 26D10, 90C23.

1. **Introduction**

Variational inequalities were introduced and studied by Stampacchia[17] in the potential theory. Variational inequalities can be regarded as natural extensions of the variational principles, the origin of which can be traced back to Euler, Lagrange and Bernoulli's brothers. The optimality conditions of the differentiable convex functions can be characterized by variational inequalities. However, it is amazing that a wide class of unrelated problems can be studied in the general and unified framework of variational inequalities, see, for example, [1, 2, 3, 4, 6, 7, **?**, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]

In recent years, the concepts of convex functions and convex sets have been generalized in various directions. Hanson[5] considered the invex functions in mathematical programming, which turned out to very interesting from practical point of view. Ben-Israiland Mond [2] introduced the concept of invex and preinvex functions. Preinvex functions may not be convex functions. They proved that the differentiable preinvex functions are invex functions, but the converse is not true. These developments were instrumental in the introduction of variational-like inequalities in 1980's. Noor [10] proved that the optimal conditions of the differentiable preinvex functions can be characterized by variational-like inequalities. Using this idea, we show that the optimality conditions of the sum of differentiable preinconvex functions and nondifferentiable strongly preinvex functions can be characterized by a class of variational-like inequalities, which is Lemma 2.1. This result motivated us to introduce strongly mixed variational-like inequalities. To the best of our knowledge, such type of strongly mixed variational-like inequalities have been not considered in the literature. It is known that the projection methods and resolvent methods can not be used to study the existence of the solution of strongly mixed variational-like inequalities involving the bifunction

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$\eta(.,.)$ and nonlinear terms. This difficulty can be overcome to use the auxiliary principle technique, which is mainly due to Lions and Stampachhia[8] as developed by Glowinski et al [4] and Noor [13, 14, 15]. We use the auxiliary principle technique to study the existence of a solution of the strongly mixed variational-like inequalities. Some iterative methods are suggested. Convergence analysis of these methods is proved under some mild conditions. Several special cases are considered, which can be obtained from main results. We expect that the ideas and techniques of this paper may stimulate further research. The comparison of these new suggested methods with other methods is an interesting problem for future research.

2. **Formulations and basic facts**

Let H be a real Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ respectively. We assume that the bifunction$\eta(.,.) : K_\eta \times K_\eta \longrightarrow H$ satisfies the condition that

$$\eta(u,v) = -\eta(v,u), \quad \forall u,v \in K_\eta,$$

unless otherwise specified.

We now recall some basic definitions and results.

Definition 2.1. *Let K_η be any set in H. The set K_η is said to be an invex set, if there exists a bifunction $\eta(.,.)$ such that*

$$u + t\eta(v,u) \in K_\eta, \quad \forall u,v \in K_\eta, t \in [0,1].$$

Definition 2.2. *The function $F : K_\eta \to H$ is said to be a strongly preinvex function, if there exist a constant $\mu > 0$ and a bifunction $\eta(.,.)$, such that*

$$F((u + t\eta(v,u)) \leq (1-t)F(u) + tF(u + \eta(v,u)) - \mu t(1-t)\|v-u\|^2, \quad \forall u,v \in K_\eta, t \in [0,1].$$

If $\mu = 0$, then definition 2.2 reduces to a classical preinvex function, which was introduced by Ben-Israil and Mond [2].
If the function F is differentiable, then

Theorem 2.1. *Let K_η be an invex set set and the function F be differentiable. Then*
(1) *F is strongly convex function, that is*

$$F((u + \eta(v,u)) \leq (1-t)F(u) + tF(v) - \mu t(1-t)\|v-u\|^2, \forall u,v \in K_\eta, t \in [0,1],$$

(2) $F(v) - F(u) \geq \langle F'(u), \eta(v,u)\rangle + \mu\|v-u\|^2, \quad \forall u,v \in K_\eta,$
(3) $\langle F'(u) - F'(v), \eta(v,u)\rangle \geq 2\mu\|v-u\|^2, \quad \forall u,v \in K,$
where $F'(u)$ is derivative of F at $u \in K_\eta$. Condition (3) says that the derivative $F'(\cdot)$ is strongly monotone with constant $\alpha = 2\mu > 0$.

We now consider the functional $I[v]$, defined as

$$I[v] = F(v) + \varphi(v), \forall v \in H, \tag{2.1}$$

where F is differentiable preinvex function and φ is a nondifferentiable preinvex function.

We now show that the minimum of the functional $I[v]$, defined by (2.1) can be characterized by a class of variational-like inequalities.

Theorem 2.2. *Let K_η be an invex convex set in H. Let F be a differentiable peinvex function and φ be a nondifferentiable strongly preinvex function. Then $u \in K_\eta$ is the minimum of the functional $I[v]$, if and only if, $u \in K_\eta$ satisfies*

$$\langle F'(u), \eta(v,u)\rangle + \varphi(v) - \varphi(u) \geq \mu\|v-u\|^2, \quad \forall v \in K, \tag{2.2}$$

where F' is the Frechet derivative of F at $u \in K_\eta$.

Proof. Let $u \in K_\eta$ be a minimum of $I[v]$ on K_η. Then

$$I[u] \leq I[v], \quad \forall v \in K_\eta. \tag{2.3}$$

Since K_η is an invex set, so, $\forall u, v \in K_\eta, \quad t \in [0,1], \quad v_t = u + t\eta(v,u) \in K_\eta$.
Replacing v by v_t in (2.3), we have

$$I[u] \leq I[v_t] = I[u + t\eta(v,u)],$$

which implies, using(2.1)

$$\begin{aligned} F(u) + \varphi(u) &\leq F(u + t\eta(v,u)) + \varphi(u + t\eta(v,u)) \\ &\leq F(u + t\eta(v,u)) + \varphi(u) + t\{\varphi(v) - \varphi(u)\} - \mu\|v-u\|^2. \end{aligned}$$

Dividing the resultant inequality by t and taking the limit as $t \longrightarrow 0$, we have ,

$$\langle F'(u), \eta(v,u)\rangle + \varphi(v) - \varphi(u) \geq \mu\|v-u\|^2, \quad v \in K_\eta,$$

the required result ().
Conversely, let $u \in K_\eta$ satisfy (2.2). We have to show that $u \in K_\eta$ is the minimum of $I[v]$ on the convex set K_η.
Consider

$$\begin{aligned} I[u] - I[v] &= -\{F(v)) + \varphi(v) - F(u) - \varphi(u)\} \\ &\leq -\{\langle F'(u), \eta(v,u)\rangle + \varphi(v) - \varphi(u)\} \\ &\leq -\mu\|v-u\|^2 \leq 0, \end{aligned} \tag{2.4}$$

where we have used the fact that F is differentiable preinvex function and (2.2). This shows that $u \in K_\eta$ is the minimum of the functional $I[v]$, defined by (2.1). □

The inequality (2.2) is called the strongly mixed variational-like inequality. This shows that the variational-like inequalities arise naturally in connection with the minimization of the differentiable preinvex functions subject to certain constraints.

First of all, we recall the following concepts. To obtain the main results, we recall some well-known concepts and results.

Definition 2.3. *An operator $T : H \to H$ is said to be:*
(1) *Strongly monotone, if there exist a constant $\alpha > 0$, such that*

$$\langle Tu - Tv, u - v\rangle \geq \alpha\|u-v\|^2, \quad \forall u, v \in H.$$

(2) *Lipschitz continuous, if there exist a constant $\beta > 0$, such that*

$$\|Tu - Tv\| \leq \beta\|u-v\|, \quad \forall u, v \in H.$$

(3) *Monotone, if*

$$\langle Tu - Tv, u - v\rangle \geq 0, \quad \forall u, v \in H.$$

Definition 2.4. *The bifunction $\eta(.,.) : H \times H \to H$ is said to be:*

(1) *Strongly monotone, if there exist a constant $\sigma > 0$, such that*

$$\langle \eta(u,v), u - v\rangle \geq \sigma\|u-v\|^2, \quad \forall u, v \in H.$$

(2) *Lipschitz continuous, if there exist a constant $\delta 0$, such that*

$$\|\eta(u,v)\| \leq \delta\|u-v\|, \quad \forall u, v \in H.$$

We would like to mention that if $\eta(v,u) = Tu - Tv$, for $T : H \longrightarrow H$, then Definition 2.4 reduces to Definition 2.3.

In many important applications, the variational-like inequalities do not arise as a result of minimization problems. The main motivation of this paper is to consider a more general variational-like inequality, which includes (2.2) as a special case.

To be more precise, let $T : H \to H$ be a continuous monotone nonlinear operator and let $\varphi : H \to R \cup \{+\infty\}$ be a function. We consider the problem of finding $u \in K_\eta$, such that

$$\langle Tu, \eta(v,u)\rangle + \varphi(v) - \varphi(u) \geq \mu\|v-u\|^2, \quad \forall v \in K_\eta, \tag{2.5}$$

which is called the strongly mixed variational-like inequalities. A wide class of problems arising in pure and applied sciences can be studied via strongly mixed variational-like inequalities (2.5).

We would like to point out that if $\eta(v,u) = v-u$, then the strongly preinvex functions become convex functions. Consequently, the strongly mixed variational-like inequalities (2.5) reduce to the strongly mixed variational inequalities, that is, find $u \in K$, a convex set in H, such that

$$\langle Tu, v-u\rangle + \varphi(v) - \varphi(u) \geq \mu\|v-u\|^2, \quad \forall v \in K, \tag{2.6}$$

which is known as the strongly mixed variational inequality. For the applications, motivation and other aspects of strongly mixed variational inequalities, see [1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] and the references therein.

Due to the presence of the bifunction $\eta(.,.)$ and nonlinear form, the projection and resolvent operator methods can not be used to discuss the existence of a solution and propose iterative methods for solving the strongly mixed variational-like inequalities. To overcome this difficulty, we use auxiliary principle technique, which is mainly due to Lions and Stamapcchia [8] and Glowinski et al [4].

Theorem 2.3. *Let T be strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, respectively. If the bifunction is strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\delta > 0$, respectively, then, for constant $\rho > 0$, such that*

$$0 < \rho < \frac{2(\alpha + 2\mu - \beta\nu)}{\beta^2 - (2\mu - \beta\nu)^2}, \quad \rho < \frac{1}{\beta\nu - 2\mu}, \tag{2.7}$$

$$\nu = 1 - 2\sigma + \delta^2. \tag{2.8}$$

then there exists a solution $u \in K_\eta$ satisfying the variational-like inequality (2.1).

Proof. We use the auxiliary principle technique to prove the existence of a solution of (2.5). To be more precise, for a given $u \in K_\eta$ satisfying (2.5), consider the problem of finding $w \in K_\eta$ such that

$$\langle \rho Tu, \eta(v,w)\rangle + \langle w-u, v-w\rangle + \rho\varphi(v) - \rho\varphi(w) - \rho\mu\|v-w\|^2 \geq 0, \forall v \in K_\eta. \tag{2.9}$$

which is called the auxiliary strongly mixed variational inequality. This technique enables us to define the mapping connecting the solution of both the problems. In this case, one has to show that the mapping connecting the solutions is a contraction mapping and consequently, it has a fixed point satisfying the original problem. For $w_1 \neq w_2 \in K_\eta$(corresponding to $u_1 \neq u_2$,), we have

$$\langle \rho Tu_1, \eta(v,w_1)\rangle + \langle w_1-u_1, v-w_1\rangle + \rho\varphi(v) - \rho\varphi(w_1) - \rho\mu\|v-w_1\|^2 \geq 0, \forall v \in K_\eta. \tag{2.10}$$

and

$$\langle \rho Tu_2, \eta(v, w_2)\rangle + \langle w_2 - u_2, v - w_2\rangle + \rho\varphi(v) - \rho\varphi(w_2) - \rho\mu\|v - w_2\|^2 \geq 0, \forall v \in K_\eta. \tag{2.11}$$

Taking $v = w_2$ in (2.10) and $v = w_1$ in (2.11) and adding the resultant, we have

$$\langle \rho(Tu_1 - Tu_2), \eta(w_2, w_1)\rangle + \langle w_1 - w_2 - (u_1 - u_2), w_2 - w_1\rangle - 2\mu\rho\|w_2 - w_1\|^2 \geq 0, \tag{2.12}$$

from which, it follows that

$$\begin{aligned} \langle w_1 - w_2, w_1 - w_2\rangle &\leq \langle u_1 - u_2 - \rho(Tu_1 - Tu_2), w_1 - w_2\rangle \\ &\quad -\langle \rho(Tu_1 - Tu_2, w_1 - w_2 - \eta(w_1, w_2)\rangle \\ &\quad -2\mu\rho\|w_1 - w_2\|^2. \end{aligned} \tag{2.13}$$

Thus, we have

$$\begin{aligned} \|w_1 - w_2\|^2 &\leq \|u_1 - u_2 - \rho(Tu_1 - Tu_2)\|\|w_1 - w_2\| \\ &\quad +\|\rho(Tu_1 - Tu_2\|\|w_1 - w_2 - \eta(w_1, w_2)\| \\ &\quad -2\rho\mu\|w_1 - w_2\|^2. \end{aligned} \tag{2.14}$$

Since the operator T is a strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, respectively, so

$$\begin{aligned} \|u_1 - u_2 - \rho(Tu_1 - Tu_2\|^2 &= \langle u_1 - u_2 - \rho(Tu_1 - Tu_2, u_1 - u_2 - \rho(Tu_1 - Tu_2\rangle \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u_1 - u_2\|^2. \end{aligned} \tag{2.15}$$

Similarly, using the strongly monotonicity and Lipschitz continuity of the bifunction $\eta(.,.)$ with constants $\sigma > 0$ and $\delta > 0$, respectively, we have

$$\begin{aligned} \|w_1 - w_2 - \eta(w_1, w_2)\|^2 &\leq \{1 - 2\sigma + \delta^2\}\|w_1 - w_2\|^2 \\ &= \nu\|w_1 - w_2\|, \end{aligned} \tag{2.16}$$

where ν is defined by (2.8).

From (2.14),(2.15) and (2.16), we have

$$\|w_1 - w_2\| \leq \{\sqrt{1 - 2\rho\alpha + \rho^2\beta^2} + \rho\beta\nu\}\|u_1 - u_2\| - 2\rho\mu\|w_1 - w_2\|, \tag{2.17}$$

which implies that

$$\begin{aligned} \|w_1 - w_2\| &\leq \frac{\sqrt{1 - 2\rho\alpha + \rho^2\beta^2} + \rho\beta\nu}{1 + 2\rho\mu}\|u_1 - u_2\| \\ &= \theta\|u_1 - u_2\|, \end{aligned} \tag{2.18}$$

where

$$\theta = \frac{\sqrt{1 - 2\rho\alpha + \rho^2\beta^2} + \rho\beta\nu}{1 + 2\rho\mu}.$$

From (2.7), it follows that $\theta < 1$. This implies that the mapping w defined by (2.9) is a contraction and consequently has a fixed point $w = u \in K_\eta$ satisfying the variational-like inequality (2.5).

□

Remark 2.1. *We note that the auxiliary problem (2.9) is equivalent to finding the minimum of the functional I[w] on the convex set K, where*

$$I[w] = \frac{1}{2}\langle w - u, w - u\rangle - \langle \rho Tu, w - u\rangle - \rho\varphi(w),$$

The function $I[u]$ is known as the gap (merit) function associated with the variational-like inequality (2.5). This equivalence can be used to suggest and analyze a number of iterative methods for solving variational-like inequalities and nonlinear programming, see, for example, Patriksson [16].

Remark 2.2. *It is clear that if $\eta(v,u) = v - u$, then the problem (2.9) is is equivalent to finding $u \in K$ such that*

$$\langle w, v-w)\rangle \geq \langle u, v-w\rangle - \rho\langle Tu, v-w\rangle - \rho\phi(v) + \rho\phi(u) - \rho\mu\|v-w\|^2, \quad \forall v \in K, \tag{2.19}$$

which is the auxiliary mixed variational inequality problem associated with strongly mixed variational inequality. Using the technique of Theorem 2.3, one can easily discuss the existence of the solution of the problem (2.6).

It is clear that, if $w = u$, then w is the solution of the strongly mixed variational-like inequality (2.5). This simple observation enables us to suggest the following iterative method.

Algorithm 2.1. *For a given u_0, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}&\langle \rho Tu_n, \eta(v, u_{n+1})\rangle + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \\ &+\rho\phi(v) - \rho\phi(u_{n+1}) \geq \rho\mu\|v - u_{n+1}\|^2, \forall v \in K_\eta,\end{aligned}$$

which is known as the explicit method.

We again use the auxiliary principle technique to suggest an implicit method for solving the strongly mixed variational-like inequalities.

For a given $u \in K_\eta$ satisfying (2.5), consider the problem of finding $w \in K_\eta$ such that

$$\langle \rho Tw, \eta(v,w)\rangle + \langle w - u, v - w\rangle + \rho\varphi(v) - \rho\varphi(w) - \rho\mu\|v-w\|^2 \geq 0, \forall v \in K_\eta. \tag{2.20}$$

which is called the auxiliary strongly mixed variational-like inequality. We would like to remark that the auxiliary principle (2.9) is quite different than the auxiliary principle (2.20).

It is clear that, if $w = u$, then w is a solution of (2.5). This fact allows to suggest the following iterative method.

Algorithm 2.2. *For a given u_0, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}\langle \rho Tu_{n+1}, \eta(v, u_{n+1})\rangle \quad + \quad &\langle u_{n+1} - u_n, v - u_{n+1}\rangle \\ &+\rho\phi(v) - \rho\phi(u_{n+1}) \geq \rho\mu\|v - u_{n+1}\|^2, \forall v \in K_\eta.\end{aligned} \tag{2.21}$$

Algorithm 3.2 is called the implicit method. We use the technique of Noor [13] to prove the convergence of Algorithm 3.2.

Theorem 2.4. *Let the operator T be monotone. If $u \in K_{\eta|}$ be solution of (2.5) and u_{n+1} is the approximate solution obtained from(2.20), then*

$$(1 + 4\rho\mu)\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2. \tag{2.22}$$

Proof. Let $u \in K_\eta$ be solution of (2.5). Then, using the monotonicity of T, we have

$$\langle Tv, \eta(v,u)\rangle + \phi(v) - \phi(u) \geq \mu\|v-u\|^2, \quad \forall v \in K_\eta. \tag{2.23}$$

Taking $v = u_{n+1}$ in (2.23), we have

$$\langle Tu_{n+1}, \eta(u_{n+1}, u)\rangle + \phi(u_{n+1}) - \phi(u) \geq \mu\|u_{n+1} - u\|^2, \quad \forall v \in K_\eta. \tag{2.24}$$

Letting $v = u$ in (2.24), we have

$$\begin{aligned}\langle \rho Tu_{n+1}, \eta(u, u_{n+1})\rangle \quad + \quad &\langle u_{n+1} - u_n, u - u_{n+1}\rangle \\ &+\rho\phi(u) - \rho\phi(u_{n+1}) \geq \rho\mu\|u - u_{n+1}\|^2, \\ &\forall v \in K_\eta.\end{aligned} \tag{2.25}$$

From (2.24) and (2.25), we obtain

$$\langle u_{n+1} - u_n, u - u_{n+1}\rangle \geq 2\mu\rho\|u - u_{n+1}\|^2. \tag{2.26}$$

Using $\forall a, b \in H, 2\langle a, b\rangle = \|a + b\|^2 - \|a\|^2 - \|b\|^2$, we obtain

$$(1 + 4\rho\mu)\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2,$$

the required result (2.20). □

Theorem 2.5. *Let H be a finite dimensional space. If u_{n+1} is the approximate solution obtained from (2.20) and $\hat{u} \in K_\eta$ is solution of (2.1), then* $\lim_{n\longrightarrow\infty} u_n = \hat{u}$.

Proof. Let $\hat{u} \in K_\eta$ be a solution of (2.1). From (2.22), it follows that the sequence $\{\|u_n - \hat{u}\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Also from (2.22), we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leq \|u_0 - \hat{u}\|^2,$$

which implies that

$$\lim_{n\longrightarrow\infty} \|u_{n+1} - u_n\| = 0. \tag{2.27}$$

Let $\bar{u}$ be a cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}$ of the sequence $\{u_n\}$ converge to $\bar{u} \in K_eta$. Replacing u_n by u_{n_j} in (2.21) and taking the limit as $n_{n_j} \longrightarrow \infty$ and using (2.27), we have

$$\langle T\hat{u}, \eta(v, \hat{u})\rangle + \phi(v) - \phi(\hat{u}) \geq \mu\|v - \hat{u}\|^2, \quad \forall v \in K_\eta,$$

which shows that $\bar{u}$ is a solution of (2.1) and

$$(1 + 4\rho\mu)\|u_{n+1} - u_n\|^2 \leq \|u_n - \hat{u}\|^2.$$

Thus it follows from the above inequality that sequence $\{u_n\}$ has exactly one cluster point $\bar{u}$ and $lim_{n\longrightarrow\infty} u_n = \bar{u}$. □

Conclusion

In this paper, we have introduced and considered a strongly mixed variational-like inequalities. It has been shown that a minimum of a sum of differentiable preinvex functions and nondifferentiable strongly preinvex can be characterized by a class of mixed variational-like inequalities. Using the auxiliary principle technique, we have discussed the existence of a solution of strongly mixed variational-like inequalities. Some iterative methods have been proposed and their convergence is investigated. Some special cases which can be obtained from our results are considered. The ideas and techniques of this paper may inspired further research in this field.

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