A GENERALIZED COMMON FIXED POINTS FOR MULTIVALUED MAPPINGS IN G_b-METRIC SPACES WITH AN APPLICATION

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In this work we are interested to prove a general common fixed point theorem for a pair of multivalued mappings satisfying a new type relation in G_b-metric spaces. The results in this paper generalize the results obtained in [3], [5],[12]. *An example and application integral equation are given to illustrate the usability of the main results.*

Keywords: Metric space, G_b-metric space, implicit relation, fixed point, multivalued maps.

MSC2010: 54H25, 47H10.

1. Introduction and Preliminary

In analysis, the fixed point theorems turn out to be very useful tools in mathematics, especially in solving differential and functional equations. In 1922, Banach[4] proved that each contraction map in a complete metric space has a unique fixed point. Mustafa and Sims[13] introduced a new notion of generalized metric space called a G-metric space. Mustafa, Sims and others studied fixed point theorems for mappings satisfying different contractive conditions [[2],[5],[12],[13],[17],[21],[22]]. Aghajani et al., in[1], extended the notion of G-metric space to the concept of G_b-metric space. Several authors have introduced a new class of generalized metric space, obtained several results in fixed point theory, (see [1]-[22]).

As a consequence of our work we obtain some results known in the case of multi-valued mappings that we will point out, and we give an application for an integral equation.

Let X be a G_b-metric space. we shall denote $B(X)$ the set of nonempty closed bounded subsets of X. Let $H_{G_b}(.,.,.)$ be the Hausdorff G_b-distance on $B(X)$, in [6] Kaewcharoen and Kaewkhao defined Hausdorf G-distance as, for $A, B, C \in B(X)$ we have

$$H_G(A,B,C) = \max\{\sup_{x\in A} G(x,B,C), \sup_{x\in B} G(x,A,C), \sup_{x\in C} G(x,B,A)\}.$$

We also in this work define Hausdorff G_b-distance on $B(X)$ as follows, for $A, B, C \in B(X)$ we have

$$H_{G_b}(A,B,C) = \max\{\sup_{x\in A} G_b(x,B,C), \sup_{x\in B} G_b(x,A,C), \sup_{x\in C} G_b(x,B,A)\}$$

where

$$\begin{aligned} G_b(x,B,C) &= d_{G_b}(x,B) + d_{G_b}(B,C) + d_{G_b}(x,C), \quad d_{G_b}(x,B) = \inf\{d_{G_b}(x,y), y \in B\}, \\ d_{G_b}(B,C) &= \inf\{d_{G_b}(x,y), x \in B, y \in C\}, \quad G_b(x,y,C) = \inf\{G_b(x,y,z), z \in C\}. \end{aligned}$$

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Note that (see ([3])) $d_{G_b}(x,y)$ is given as $d_{G_b}(x,y) = G_b(x,y,y) + G_b(x,x,y)$ which defines a b-metric on X. A mapping $T : X \longrightarrow B(X)$ is called a multivalued mapping. A point $x \in X$ is called a fixed point of T if $x \in Tx$.

Definition 1.1 ([3]). *Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that G_b: $X \times X \times X \longrightarrow \mathbb{R}^+$ satisfies:*
(G_{b1}) $G_b(x,y,z) = 0 \quad if \quad x = y = z,$
(G_{b2}) $0 < G_b(x,x,y),$ *for all* $x, y \in X,$ *with* $x \neq y,$
(G_{b3}) $G_b(x,x,y) \leqslant G_b(x,y,z),$ *for all* $x,y,z \in X,$ *with* $z \neq y,$
(G_{b4}) $G_b(x,y,z) = G_b(x,z,y) = G_b(y,x,z) = \ldots$ *(symmetry in all three variables), and*
(G_{b5}) $G_b(x,y,z) \leqslant s(G_b(x,a,a) + G_b(a,y,z)), \forall x,y,z,a \in X$ *(triangle inequality).*
Then G_b is called a generalized G-metric and the pair (X, G_b) is called a G_b-metric space.

It is clear that if $s = 1$, then (X, G) becomes a G-metric space.

Example 1.1 ([1]). *Let (X,G) be a G-metric space. Consider $G_b(x,y,z) = (G(x,y,z))^p$, where $p > 1$ is a real number. Then, G_b is a G_b-metric with $s = 2^{p-1}$*

Definition 1.2 ([1]). *Let (X, G_b) be a G_b-metric space. A sequence (x_n) in X is said to be*
(1) *G_b-Cauchy sequence if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $m,n,l \geq n_0$, $G_b(x_n,x_m,x_l) < \varepsilon$.*
(2) *G_b-convergent to a point $x \in X$ if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $m,n \geq n_0$, $G_b(x_n,x_m,x) < \varepsilon$.*

Proposition 1.1 ([3]). *Let (X, G_b) be a G_b-metric space. Then, the following are equivalent:*
(1) *the sequence (x_n) is G_b-Cauchy,*
(2) $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ *such that,* $\forall m,n \geq n_0$, $G_b(x_n,x_m,x_m) < \varepsilon$.

Proposition 1.2 ([3]). *Let (X, G_b) be a G_b-metric space. Then, the following are equivalent:*
(1) *(x_n) is G_b-convergent to x.*
(2) $\lim\limits_{n\to+\infty} G_b(x_n,x_n,x) = 0.$
(3) $\lim\limits_{n\to+\infty} G_b(x_n,x,x) = 0.$
(4) $\lim\limits_{n,m\to+\infty} G_b(x_n,x_m,x) = 0.$

Definition 1.3 ([1]). *A G_b-metric space X is called G_b-complete if every G_b-Cauchy sequence is G_b-convergent in X.*

Definition 1.4 ([14]). *Let (X, G_b) and (X', G_b') be two G_b-metric spaces. Then a function $T : X \longrightarrow X'$ is G_b-continuous at a point $x \in X$ if and only if it is G_b-sequentially continuous at x, that is, whenever (x_n) is G_b-convergent to x, (Tx_n) is G_b'-convergent to Tx.*

Mustafa and Sims proved that each G-metric function $G(x,y,z)$ is jointly continuous in all three of its variables (see, Proposition 8 [11]). But in general a G_b-metric function $G_b(x,y,z)$ for $s > 1$ is not jointly continuous in all three of its variables. Now we recall an example of a discontinuous G_b-metric.

Example 1.2. *Let $X = \mathbb{N} \bigcup \infty$ and let $d : X \times X \longrightarrow \mathbb{R}^+$ and $s = 3$ such that:*

$$d(n,m) = \begin{cases} 0 & if \quad m = n \\ \mid \frac{1}{m} - \frac{1}{n} \mid & si \quad m,n \text{ are even or } mn = \infty \\ 5 & if \quad m,n \text{ are odd and } m \neq n \\ 2 & otherwise. \end{cases}$$

Then it is easy to see that for all $m,n,p \in X$, we have

$$d(m,p) \leq 3(d(m,n) + d(n,p)).$$

Thus (X,d) is a b-metric space with $s = 3$. Let $G_b(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}$. It is easy to see that G_b is a G_b-metric with $s = 3$. Now, we show that $G_b(x,y,z)$ is not a continuous function. Indeed:
suppose that G_b is continuous, take $x_n = 2n$ and $y_n = z_n = 1$, then we have $x_n \longrightarrow \infty$, $y_n \longrightarrow 1$ and $z_n \longrightarrow 1$. The application: $T{:}X \longrightarrow \mathbb{R}^+$ with $Tx = G_b(x,1,1)$ will be continuous, but we have :

$$\begin{aligned} \mid T_{x_n} - T_\infty \mid &= \mid G_b(x_n,1,1) - G_b(\infty,1,1) \mid \\ &= 1 \nrightarrow 0 \end{aligned}$$

which is a contradiction.

2. Main results

Definition 2.1. *Let $s \geq 1$, and $\mathcal{F}_s$ be the set of all functions $F(t_1,t_2,...,t_{11}) : \mathbb{R}_+^{11} \longrightarrow \mathbb{R}$ such that:*
$(\mathcal{F}_1)$: F is continuous in variables t_1, t_2, t_3, t_6, t_7, t_{10}, t_{11} and non increasing in variables t_3, t_4,..., t_{11}.
$(\mathcal{F}_2)$: $\exists h_1 \in [0,1[$, such that $\forall u,v \geq 0$:
$F(u,v,sv,u,u,s(v{+}u),0,u,u,0,s(u{+}v)) \leq 0$ or $F(u,u,0,0,0,u,v,0,0,v,u) \leq 0 \Rightarrow u \leq h_1 v$.
$(\mathcal{F}_3)$ $\exists h_2 \in \left[0, \frac{1}{s}\right[$, such that $\forall u,v \geq 0$: $F(u,0,0,v,v,sv,0,v,v,0,sv) \leq 0 \Rightarrow u \leq h_2 v$.
Moreover, if $\exists\, \alpha, \beta \geq 0$ such that $\forall u,v,w \geq 0$:
$F(u,v,0,s(u+w),s(u+w),u,w,s(u+w),s(u+w),w,u) \leq 0 \Rightarrow u \leqslant \alpha v + \beta w$, we say that F check $(\mathcal{F}_4)$.

Example 2.1. *$F(t_1,t_2,...,t_{11}) = t_1 - \lambda \max\{t_3,t_4,t_5\}$ with $0 \leq \lambda < \frac{1}{s}$.*

Example 2.2. *$F(t_1,t_2,...,t_{11}) = t_1 - \gamma(t_2)t_2$ with $\gamma : [0,\infty) \longrightarrow [0,\frac{1}{s})$.*

Example 2.3. *$F(t_1,t_2,...,t_{11}) = t_1 - \lambda \max\{t_2,t_3,t_4,t_5,t_6,t_8,t_{10}\}$ with $0 \leq \lambda < \frac{1}{s^2+s}$.*

Example 2.4. *$F(t_1,t_2,...,t_{11}) = t_1 - \lambda(t_6 + t_7)$ with $0 \leq \lambda < \frac{1}{s^2+s}$.*

Example 2.5. *$F(t_1,t_2,...,t_{11}) = t_1 - \lambda \max\{t_4 + t_6, t_7 + t_{10}\}$ with $0 \leq \lambda < \frac{1}{s^2+s+1}$.*

Example 2.6. *$F(t_1,t_2,...,t_{11}) = t_1 - \lambda \max\{t_6 + t_7, t_8 + t_9, t_{10} + t_{11}\}$ with $0 \leq \lambda < \frac{1}{s^2+s}$.*

Example 2.7. *$F(t_1,t_2,...,t_{11}) = t_1 - \lambda \max\{t_6 + t_9, t_7 + t_{10}, t_8 + t_{11}\}$ with $0 \leq \lambda < \frac{1}{s^2+s+1}$.*

Theorem 2.1. *Let (X,G_b) be a complete G_b-metric space with coefficient $s \geq 1$, $T,R : X \longrightarrow B(X)$ such that $\forall x,y,z \in X$*

$$F\begin{pmatrix} H_{G_b}(Tx,Ry,Rz), G_b(x,y,z), G_b(x,Tx,Tx), \\ G_b(y,Ry,Ry), G_b(z,Rz,Rz), G_b(x,Ry,Ry), \\ G_b(y,Tx,Tx), G_b(y,Rz,Rz), G_b(z,Ry,Ry), \\ G_b(z,Tx,Tx), G_b(x,Rz,Rz) \end{pmatrix} \leq 0, \tag{1}$$

and

$$F\begin{pmatrix} H_{G_b}(Rx,Ty,Tz), G_b(x,y,z), G_b(x,Rx,Rx), \\ G_b(y,Ty,Ty), G_b(z,Tz,Tz), G_b(x,Ty,Ty), \\ G_b(y,Rx,Rx), G_b(y,Tz,Tz), G_b(z,Ty,Ty), \\ G_b(z,Rx,Rx), G_b(x,Tz,Tz) \end{pmatrix} \leq 0, \tag{2}$$

with $F \in \mathcal{F}_s$. Then, T and R have a common fixed point $x \in X$. Moreover, if x is absolutely fixed for T and R (which means that $Tx = Rx = \{x\}$), then the fixed point is unique.

For the proof of this theorem we need two lemmas.

Lemma 2.1. *Let (X, G_b) be a G_b-metric space with $s \geq 1$ and $A, B \in B(X)$. Then for each $a \in A$, we have*

$$G_b(a, B, B) \leq H_{G_b}(A, B, B).$$

Proof.

$$H_{G_b}(A, B, B) \geq \sup_{x \in A} G_b(x, B, B) \geq G_b(a, B, B).$$

Lemma 2.2. *Let (X, G_b) be a G_b-metric space with $s \geq 1$. If $A, B \in B(X)$ and $x \in A$, then for each $\varepsilon > 0$, there exists $y \in B$ such that*

$$G_b(x, y, y) \leq H_{G_b}(A, B, B) + \varepsilon.$$

Proof.

By Lemma 2.1 and the characterization of inf we have, for each $\varepsilon > 0$, there exists $y \in B$ such that

$$\begin{aligned} G_b(x, y, y) &\leq \inf\{G_b(x, z, z), z \in B\} + \varepsilon \\ &\leq 2 \inf_{z \in B}(G_b(x, z, z) + G_b(z, x, x)) + \varepsilon \\ &= 2d_{G_b}(x, B) + \varepsilon = G_b(x, B, B) + \varepsilon \\ &\leq H_{G_b}(A, B, B) + \varepsilon. \end{aligned}$$

Proof of Theorem 2.1 .

Existence.

For $x_0 \in X$, and $x_1 \in Tx_0$. According to (1), with $x = x_0$ and $y = z = x_1$ we have

$$F\begin{pmatrix} H_{G_b}(Tx_0, Rx_1, Rx_1), G_b(x_0, x_1, x_1), G_b(x_0, Tx_0, Tx_0), \\ G_b(x_1, Rx_1, Rx_1), G_b(x_1, Rx_1, Rx_1), G_b(x_0, Rx_1, Rx_1), \\ G_b(x_1, Tx_0, Tx_0), G_b(x_1, Rx_1, Rx_1), G_b(x_1, Rx_1, Rx_1), \\ G_b(x_1, Tx_0, Tx_0), G_b(x_0, Rx_1, Rx_1) \end{pmatrix} \leq 0. \quad (3)$$

According to (G_{b5}) we have:

$$\begin{aligned} G_b(x_0, Tx_0, Tx_0) &\leqslant s(G_b(x_0, x_1, x_1) + G_b(x_1, Tx_0, Tx_0)) \\ &= sG_b(x_0, x_1, x_1), \end{aligned}$$

by Lemma 2.1 and (G_{b5}) we have :

$$\begin{aligned} G_b(x_0, Rx_1, Rx_1) &\leqslant s(G_b(x_0, x_1, x_1) + G_b(x_1, Rx_1, Rx_1)) \\ &\leqslant s(G_b(x_0, x_1, x_1) + H_{G_b}(Tx_0, Rx_1, Rx_1)). \end{aligned}$$

By Lemma 2.1, and $(\mathcal{F}_1)$, (3) becomes :

$$F\begin{pmatrix} H_{G_b}(Tx_0, Rx_1, Rx_1), G_b(x_0, x_1, x_1), sG_b(x_0, x_1, x_1), H_{G_b}(Tx_0, Rx_1, Rx_1), \\ H_{G_b}(Tx_0, Rx_1, Rx_1), s(G_b(x_0, x_1, x_1) + H_{G_b}(Tx_0, Rx_1, Rx_1)), \\ 0, H_{G_b}(Tx_0, Rx_1, Rx_1), H_{G_b}(Tx_0, Rx_1, Rx_1), \\ 0, s(G_b(x_0, x_1, x_1) + H_{G_b}(Tx_0, Rx_1, Rx_1)) \end{pmatrix} \leq 0. \quad (4)$$

According to $(\mathcal{F}_2)$: $\exists h_1 \in [0, 1[$, such that: $H_{G_b}(Tx_0, Rx_1, Rx_1) \leqslant h_1\, G_b(x_0, x_1, x_1)$.

Now, by Lemma 2.2 we have for $\varepsilon = \frac{1}{2}(1 - h_1)G_b(x_0, x_1, x_1)$, there exists $x_2 \in Rx_1$ such that:

$$\begin{aligned} G_b(x_1, x_2, x_2) &\leqslant H_{G_b}(Tx_0, Rx_1, Rx_1) + \varepsilon \\ &\leqslant h_1 G_b(x_0, x_1, x_1) + \frac{1}{2}(1 - h_1)G_b(x_0, x_1, x_1) \\ &= \frac{1}{2}(1 + h_1)G_b(x_0, x_1, x_1) = hG_b(x_0, x_1, x_1) \text{ with } h = \frac{1}{2}(1 + h_1) < 1. \end{aligned}$$

According to (2), with $x = x_1$ and $y = z = x_2$ we have :

$$F\begin{pmatrix} H_{G_b}(Rx_1, Tx_2, Tx_2), G_b(x_1, x_2, x_2), G_b(x_1, Rx_1, Rx_1), \\ G_b(x_2, Tx_2, Tx_2), G_b(x_2, Tx_2, Tx_2), G_b(x_1, Tx_2, Tx_2), \\ G_b(x_2, Rx_1, Rx_1), G_b(x_2, Tx_2, Tx_2), G_b(x_2, Tx_2, Tx_2), \\ G_b(x_2, Rx_1, Rx_1), G_b(x_1, Tx_2, Tx_2) \end{pmatrix} \leq 0. \tag{5}$$

By Lemma 2.1, (G_{b5}) and $(\mathcal{F}_1)$ (5) becomes :

$$F\begin{pmatrix} H_{G_b}(Rx_1, Tx_2, Tx_2), G_b(x_1, x_2, x_2), sG_b(x_1, x_2, x_2), H_{G_b}(Rx_1, Tx_2, Tx_2), \\ H_{G_b}(Rx_1, Tx_2, Tx_2), s(G_b(x_1, x_2, x_2) + H_{G_b}(Rx_1, Tx_2, Tx_2)), \\ 0, H_{G_b}(Rx_1, Tx_2, Tx_2), H_{G_b}(Rx_1, Tx_2, Tx_2), \\ 0, s(G_b(x_1, x_2, x_2) + H_{G_b}(Rx_1, Tx_2, Tx_2)) \end{pmatrix} \leq 0. \tag{6}$$

According to $(\mathcal{F}_2)$: $\exists h_1 \in [0,1[$, such that $H_{G_b}(Rx_1, Tx_2, Tx_2) \leqslant h_1\, G_b(x_1, x_2, x_2)$.

By Lemma 2.2 we have for $\varepsilon = \frac{1}{2}(1-h_1)G_b(x_1, x_2, x_2)$, there exists $x_3 \in Tx_2$ such that:

$$\begin{aligned} G_b(x_2, x_3, x_3) &\leqslant H_{G_b}(Rx_1, Tx_2, Tx_2) + \varepsilon \\ &\leqslant h_1 G_b(x_1, x_2, x_2) + \frac{1}{2}(1-h_1)G_b(x_1, x_2, x_2) \\ &= \frac{1}{2}(1+h_1)G_b(x_1, x_2, x_2) = hG_b(x_1, x_2, x_2) \text{ with } h = \frac{1}{2}(1+h_1) < 1. \end{aligned}$$

By recurrence, we construct a sequence (x_n) with $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Rx_{2n+1}$ which satisfies:

$$G_b(x_{2n+1}, x_{2n+2}, x_{2n+2}) \leqslant h\ \ G_b(x_{2n}, x_{2n+1}, x_{2n+1}) \tag{7}$$

and

$$G_b(x_{2n}, x_{2n+1}, x_{2n+1}) \leqslant h\ \ G_b(x_{2n-1}, x_{2n}, x_{2n}) \tag{8}$$

According to (7) and (8) for everything $n \in \mathbb{N}^*$, we have

$$G_b(x_n, x_{n+1}, x_{n+1}) \leqslant h\ \ G_b(x_{n-1}, x_n, x_n)$$

from where

$$G_b(x_n, x_{n+1}, x_{n+1}) \leqslant h^n\ \ G_b(x_0, x_1, x_1).$$

Now we have to show that (x_n) is a Cauchy sequence. Let $m, n \in \mathbb{N}^*$, then

$$\begin{aligned} G_b(x_n, x_{n+m}, x_{n+m}) \leq\ & sG_b(x_n, x_{n+1}, x_{n+1}) + s^2 G_b(x_{n+1}, x_{n+2}, x_{n+2}) + \\ & s^3 G_b(x_{n+2}, x_{n+3}, x_{n+3}) + ... + s^{m-1} G_b(x_{n+m-1}, x_{n+m}, x_{n+m}). \end{aligned}$$

On the other hand, we have :

$$\begin{aligned} G_b(x_n, x_{n+m}, x_{n+m}) &\leq sh^n G_b(x_0, x_1, x_1) + s^2 h^{n+1} G_b(x_0, x_1, x_1) + s^3 h^{n+2} G_b(x_0, x_1, x_1) \\ &\quad +... + s^{m-1} h^{n+m-2} G_b(x_0, x_1, x_1) + s^{m-1} h^{n+m-1} G_b(x_0, x_1, x_1) \\ &\leq sh^n (1 + (sh) + (sh)^2 + ... + (sh)^{m-2} + (sh)^{m-1}) G_b(x_0, x_1, x_1) \\ &= sh^n \left(\frac{1-(sh)^m}{1-sh}\right) G_b(x_0, x_1, x_1) \leq \left(\frac{h^n s}{1-sh}\right) G_b(x_0, x_1, x_1). \end{aligned}$$

from where $\lim\limits_{n\to\infty} G_b(x_n, x_{n+m}, x_{n+m}) = 0$ for $m \in \mathbb{N}^*$. By Proposition 1.1, then (x_n) is a Cauchy sequence. As the G_b-metric space (X, G_b) is complete, there exists $x \in X$ such that $\lim\limits_{n\to\infty} G_b(x_n, x_n, x) = 0$. Next we show that $x \in Tx$ and $x \in Rx$, indeed, by (2) we have

$$F\begin{pmatrix} H_{G_b}(Rx_{2n-1}, Tx, Tx), G_b(x_{2n-1}, x, x), G_b(x_{2n-1}, Rx_{2n-1}, Rx_{2n-1}), \\ G_b(x, Tx, Tx), G_b(x, Tx, Tx), G_b(x_{2n-1}, Tx, Tx), \\ G_b(x, Rx_{2n-1}, Rx_{2n-1}), G_b(x, Tx, Tx), \\ G_b(x, Tx, Tx), G_b(x, Rx_{2n-1}, Rx_{2n-1}), G_b(x_{2n-1}, Tx, Tx) \end{pmatrix} \leq 0. \tag{9}$$

According to (G_{b5}) we have

$$G_b(x_{2n-1}, Tx, Tx) \leqslant s(G_b(x_{2n-1}, x, x) + G_b(x, Tx, Tx)),$$

$$\begin{aligned} G_b(x_{2n-1}, Rx_{2n-1}, Rx_{2n-1}) &\leqslant s(G_b(x_{2n-1}, x_{2n}, x_{2n}) + G_b(x_{2n}, Rx_{2n-1}, Rx_{2n-1})) \\ &= sG_b(x_{2n-1}, x_{2n}, x_{2n}) \end{aligned}$$

and

$$\begin{aligned} G_b(x, Rx_{2n-1}, Rx_{2n-1}) &\leqslant s(G_b(x, x_{2n}, x_{2n}) + G_b(x_{2n}, Rx_{2n-1}, Rx_{2n-1}) \\ &= sG_b(x, x_{2n}, x_{2n}). \end{aligned}$$

So by $(\mathcal{F}_1)$, (9) becomes :

$$F\begin{pmatrix} H_{G_b}(Rx_{2n-1}, Tx, Tx), G_b(x_{2n-1}, x, x), sG_b(x_{2n-1}, x_{2n}, x_{2n}), \\ G_b(x, Tx, Tx), G_b(x, Tx, Tx), s(G_b(x_{2n-1}, x, x) + G_b(x, Tx, Tx)), \\ sG_b(x, x_{2n}, x_{2n}), G_b(x, Tx, Tx), \\ G_b(x, Tx, Tx), sG_b(x, x_{2n}, x_{2n}), s(G_b(x_{2n-1}, x, x) + G_b(x, Tx, Tx)) \end{pmatrix} \leq 0. \qquad (10)$$

Letting $n \to \infty$ we obtain

$$F\begin{pmatrix} \lim\inf_{n\to\infty} H_{G_b}(Rx_{2n-1}, Tx, Tx), 0, 0, G_b(x, Tx, Tx), G_b(x, Tx, Tx), \\ sG_b(x, Tx, Tx), 0, G_b(x, Tx, Tx), G_b(x, Tx, Tx), 0, sG_b(x, Tx, Tx)) \end{pmatrix} \leq 0$$

because we have $H_{G_b}(Rx_{2n-1}, T(x_{2n}), T(x_{2n})) \leq h_1 G_b(x_{2n-1}, x_{2n}, x_{2n})$, $x_{2n+1} \in T(x_{2n})$

$$\begin{aligned} H_{G_b}(Rx_{2n-1}, Tx, Tx) &\leq s(H_{G_b}(Rx_{2n-1}, \{x_{2n+1}\}, \{x_{2n+1}\}) + H_{G_b}(\{x_{2n+1}\}, Tx, Tx)) \\ &\leq sH_{G_b}(Rx_{2n-1}, T(x_{2n}), T(x_{2n})) + s^2(H_{G_b}(\{x_{2n+1}\}, \{x\}, \{x\}) \\ &+ H_{G_b}(\{x\}, Tx, Tx)) \\ &\leq sh_1 G_b(x_{2n-1}, x_{2n}, x_{2n}) + s^2 G_b(x_{2n+1}, x, x) \\ &+ s^2 H_{G_b}(\{x\}, Tx, Tx), \end{aligned}$$

so we deduce that the sequence $(H_{G_b}(Rx_{2n-1}, Tx, Tx))_n$ is bounded.

Now, by $(\mathcal{F}_3)$, $\exists h_2 \in \left[0, \frac{1}{s}\right[$, such that:

$$\lim\inf_{n\to\infty} H_{G_b}(Rx_{2n-1}, Tx, Tx) \leqslant h_2 G_b(x, Tx, Tx). \qquad (11)$$

On the other hand we show that $G_b(x, Tx, Tx) = 0$. Suppose that $G_b(x, Tx, Tx) > 0$, then

$$\begin{aligned} G_b(x, Tx, Tx) &\leq s(G_b(x, x_{2n}, x_{2n}) + G_b(x_{2n}, Tx, Tx)) \\ &\leq s(G_b(x, x_{2n}, x_{2n}) + H_{G_b}(Rx_{2n-1}, Tx, Tx)). \end{aligned}$$

By (11) we have

$$\begin{aligned} G_b(x, Tx, Tx) &\leq \lim\inf_{n\to\infty} s(G_b(x, x_{2n}, x_{2n}) + H_{G_b}(Rx_{2n-1}, Tx, Tx)) \\ &\leq sh_2 G_b(x, Tx, Tx) < G_b(x, Tx, Tx) \end{aligned}$$

which is a contradiction, then $G_b(x, Tx, Tx) = 0$ from where $x \in Tx$.

In the same way by (1) with $x = x_{2n}$ and $y = z = x$, we find $G_b(x, Rx, Rx) = 0$, hence $x \in Rx$ and consequently T and R have a common fixed point $x \in X$.

Unicity.

Suppose that $Tx = Rx = \{x\}$ and $y \in X$ is an other common fixed point of T and R then by (1) we have

$$F\begin{pmatrix} H_{G_b}(Tx, Ry, Ry), G_b(x, y, y), G_b(x, Tx, Tx), \\ G_b(y, Ry, Ry), G_b(y, Ry, Ry), G_b(x, Ry, Ry), \\ G_b(y, Tx, Tx), G_b(y, Ry, Ry), G_b(y, Ry, Ry), \\ G_b(y, Tx, Tx), G_b(x, Ry, Ry) \end{pmatrix} \leq 0,$$

implies

$$F\begin{pmatrix} G_b(x,y,y), G_b(x,y,y), 0, 0, 0, G_b(x,y,y), \\ G_b(y,x,x), 0, 0, G_b(y,x,x), G_b(x,y,y) \end{pmatrix} \leq 0.$$

According to ($\mathcal{F}_2$) we have $G_b(x,y,y) \leq h_1 G_b(y,x,x)$, in the same way by (2) we find $G_b(y,x,x) \leq h_1 G_b(x,y,y)$. Then $G_b(x,y,y) \leq h_1^2 G_b(x,y,y)$ then $G_b(x,y,y) = 0$ from where $x = y$.

From Theorem 2.1 and $T = R$ a we obtain corollary

Corollary 2.1. *Let (X, G_b) be a complete G_b-metric space with coefficient $s \geq 1$, $T{:}X \longrightarrow B(X)$ such that $\forall x, y, z \in X.$:*

$$F\begin{pmatrix} H_{G_b}(Tx,Ty,Tz), G_b(x,y,z), G_b(x,Tx,Tx), \\ G_b(y,Ty,Ty), G_b(z,Tz,Tz), G_b(x,Ty,Ty), \\ G_b(y,Tx,Tx), G_b(y,Tz,Tz), G_b(z,Ty,Ty), \\ G_b(z,Tx,Tx), G_b(x,Tz,Tz) \end{pmatrix} \leq 0, \tag{12}$$

with $F \in (\mathcal{F}_s)$. Then T has a fixed point $x \in X$. Moreover, if x is absolutely fixed, then it is unique.

Example 2.8. *Let $X = [0, +\infty[$. Define a mapping $T{:}X \longrightarrow B(X)$ by $Tx = [0, \frac{x}{4}]$.*

Define a G_b-metric with coefficient $s = 2$ on X by

$$G_b(x,y,z) = (\mid x-y \mid + \mid y-z \mid + \mid x-z \mid)^2.$$

We prove that T check

$$H_{G_b}(Tx,Ty,Tz) \leqslant \frac{1}{9} \ \max\{G_b(x,Tx,Tx), G_b(y,Ty,Ty), G_b(z,Tz,Tz)\}$$

Indeed, we have $d_{G_b}(x,y) = G_b(x,y,y) + G_b(y,x,x) = 8(x-y)^2$ for all $x, y \in X$.

Let $x, y, z \in X$. If $x = y = z = 0$ then $H_{G_b}(Tx,Ty,Tz) = 0$. Thus we may assume that x, y and z are not all zero. Without loss of generality we assume that $x \leq y \leq z$. Then

$$H_{G_b}(Tx,Ty,Tz)) = H_{G_b}\left(\left[0,\frac{x}{4}\right],\left[0,\frac{y}{4}\right],\left[0,\frac{z}{4}\right]\right) = \max\left\{\begin{array}{c} \sup\limits_{0\leq a\leq \frac{x}{4}} G_b\left(a,\left[0,\frac{y}{4}\right],\left[0,\frac{z}{4}\right]\right), \\ \sup\limits_{0\leq b\leq \frac{y}{4}} G_b\left(b,\left[0,\frac{x}{4}\right],\left[0,\frac{z}{4}\right]\right), \\ \sup\limits_{0\leq c\leq \frac{z}{4}} G_b\left(c,\left[0,\frac{x}{4}\right],\left[0,\frac{y}{4}\right]\right) \end{array}\right\}.$$

Since $x \leq y \leq z$, then $\left[0,\frac{x}{4}\right] \subseteq \left[0,\frac{y}{4}\right] \subseteq \left[0,\frac{z}{4}\right]$ which implies that

$$d_{G_b}\left(\left[0,\frac{x}{4}\right],\left[0,\frac{y}{4}\right]\right) = d_{G_b}\left(\left[0,\frac{y}{4}\right],\left[0,\frac{z}{4}\right]\right) = d_{G_b}\left(\left[0,\frac{x}{4}\right],\left[0,\frac{z}{4}\right]\right) = 0.$$

For each $0 \leq a \leq \frac{x}{4}$ we have

$$G_b\left(a,\left[0,\frac{y}{4}\right],\left[0,\frac{z}{4}\right]\right) = d_{G_b}\left(a,\left[0,\frac{y}{4}\right]\right) + d_{G_b}\left(\left[0,\frac{y}{4}\right],\left[0,\frac{z}{4}\right]\right) + d_{G_b}\left(a,\left[0,\frac{z}{4}\right]\right) = 0.$$

Also for each $0 \leq b \leq \frac{y}{4}$ we have

$$\begin{aligned} G_b\left(b,\left[0,\frac{x}{4}\right],\left[0,\frac{z}{4}\right]\right) &= d_{G_b}\left(b,\left[0,\frac{x}{4}\right]\right) + d_{G_b}\left(\left[0,\frac{x}{4}\right],\left[0,\frac{z}{4}\right]\right) + d_{G_b}\left(b,\left[0,\frac{z}{4}\right]\right) \\ &= d_{G_b}\left(b,\left[0,\frac{x}{4}\right]\right) = \inf_{0\leq e\leq \frac{x}{4}} d_{G_b}(b,e) \\ &= \begin{cases} 0, & \text{if} \quad 0 \leq b \leq \frac{x}{4} \\ 8\left(b-\frac{x}{4}\right)^2, & \text{if} \quad \frac{x}{4} \leq b \leq \frac{y}{4} \end{cases} \end{aligned}$$

Implies

$$\sup_{0\leq b\leq \frac{y}{4}} G_b\left(b,\left[0,\frac{x}{4}\right],\left[0,\frac{z}{4}\right]\right) = \frac{1}{2}(y-x)^2.$$

Moreover, for each $0 \leq c \leq \frac{z}{4}$ *we have*

$$\begin{aligned} G_b\left(c,\left[0,\frac{x}{4}\right],\left[0,\frac{y}{4}\right]\right) &= d_{G_b}\left(c,\left[0,\frac{x}{4}\right]\right)+d_{G_b}\left(\left[0,\frac{x}{4}\right],\left[0,\frac{y}{4}\right]\right)+d_{G_b}\left(c,\left[0,\frac{y}{4}\right]\right) \\ &= d_{G_b}\left(c,\left[0,\frac{x}{4}\right]\right)+d_{G_b}\left(c,\left[0,\frac{y}{4}\right]\right) \\ &= \begin{cases} 0, & if \quad 0\leq c\leq \frac{x}{4} \\ 8\left(c-\frac{x}{4}\right)^2, & if \quad \frac{x}{4}\leq c\leq \frac{y}{4} \\ 8\left(c-\frac{x}{4}\right)^2+8\left(c-\frac{y}{4}\right)^2, & if\ \frac{y}{4}\leq c\leq \frac{z}{4} \end{cases} \end{aligned}$$

Implies

$$\sup_{0\leq c\leq \frac{z}{4}} G_b\left(c,\left[0,\frac{x}{4}\right],\left[0,\frac{y}{4}\right]\right)=\frac{1}{2}\left[(z-x)^2+(z-y)^2\right].$$

Thus we deduce that

$$\begin{aligned} H_{G_b}(Tx,Ty,Tz) &= \max\left\{0,\frac{1}{2}(y-x)^2,\frac{1}{2}\left[(z-x)^2+(z-y)^2\right]\right\}=\frac{1}{2}\left[(z-x)^2+(z-y)^2\right] \\ &\leq \frac{1}{2}\left[(z-x)^2+(z-x)^2\right]=(z-x)^2. \end{aligned}$$

So,

$$H_{G_b}(Tx,Ty,Tz)\leq (z-x)^2. \tag{13}$$

On the other hand, we have

$$\begin{aligned} G_b(x,Tx,Tx) &= d_{G_b}\left(x,\left[0,\frac{x}{4}\right]\right)+d_{G_b}\left(\left[0,\frac{x}{4}\right],\left[0,\frac{x}{4}\right]\right)+d_{G_b}\left(x,\left[0,\frac{x}{4}\right]\right) \\ &= 2d_{G_b}\left(x,\left[0,\frac{x}{4}\right]\right)=2\inf_{0\leq e\leq \frac{x}{4}} d_{G_b}(x,e)=16(x-\frac{x}{4})^2=9x^2. \end{aligned}$$

From where

$$\max\{G_b(x,Tx,Tx),G_b(y,Ty,Ty),G_b(z,Tz,Tz)\}=9z^2. \tag{14}$$

By (13) *and* (14) *we have*

$$\begin{aligned} H_{G_b}(Tx,Ty,Tz) &= (z-x)^2\leq z^2=\frac{1}{9}\times 9z^2 \\ &= \frac{1}{9}\max\{G_b(x,Tx,Tx),G_b(y,Ty,Ty),G_b(z,Tz,Tz)\} \end{aligned}$$

All the conditions of Corollary 2.1 are satisfied with F as in Example 2.1, then 0 *is the unique absolutely fixed point of T.*

If T and R are single mappings, the we obtain the following :

Theorem 2.2. *Let (X,G_b) be a complete G_b-metric space with coefficient $s\geq 1$, $T,R{:}X\longrightarrow X$ such that $\forall x,y,z\in X$. :*

$$F\begin{pmatrix} G_b(Tx,Ry,Rz),G_b(x,y,z),G_b(x,Tx,Tx), \\ G_b(y,Ry,Ry),G_b(z,Rz,Rz),G_b(x,Ry,Ry), \\ G_b(y,Tx,Tx),G_b(y,Rz,Rz),G_b(z,Ry,Ry), \\ G_b(z,Tx,Tx),G_b(x,Rz,Rz) \end{pmatrix}\leq 0, \tag{15}$$

and

$$F\begin{pmatrix} G_b(Rx,Ty,Tz),G_b(x,y,z),G_b(x,Rx,Rx), \\ G_b(y,Ty,Ty),G_b(z,Tz,Tz),G_b(x,Ty,Ty), \\ G_b(y,Rx,Rx),G_b(y,Tz,Tz),G_b(z,Ty,Ty), \\ G_b(z,Rx,Rx),G_b(x,Tz,Tz) \end{pmatrix}\leq 0, \tag{16}$$

with $F\in\mathcal{F}_s$. Then, T and R have a unique common fixed point.

Moreover, if F check $(\mathcal{F}_4)$, then T and R are G_b-continuous at this point.

Proof.

The unique common fixed point of T and R becomes from Theorem 2.1.

The continuity.

Suppose that F check $(\mathcal{F}_4)$, let (y_n) is G_b-convergent to x. For all $n \in \mathbb{N}$, by (2) we have

$$F\begin{pmatrix} G_b(Rx,Ty_n,Ty_n),G_b(x,y_n,y_n),G_b(x,Rx,Rx),G_b(y_n,Ty_n,Ty_n), \\ G_b(y_n,Ty_n,Ty_n),G_b(x,Ty_n,Ty_n),G_b(y_n,Rx,Rx),G_b(y_n,Ty_n,Ty_n), \\ ,G_b(y_n,Ty_n,Ty_n),G_b(y_n,Rx,Rx),G_b(x,Ty_n,Ty_n) \end{pmatrix} \leq 0.$$

As $\quad G_b(y_n,Ty_n,Ty_n) \leqslant s(G_b(y_n,x,x)+G_b(x,Ty_n,Ty_n))$. We have

$$F\begin{pmatrix} G_b(x,Ty_n,Ty_n),G_b(x,y_n,y_n),0,s(G_b(y_n,x,x)+G_b(x,Ty_n,Ty_n)), \\ s(G_b(y_n,x,x)+G_b(x,Ty_n,Ty_n)),G_b(x,Ty_n,Ty_n),G_b(y_n,x,x), \\ s(G_b(y_n,x,x)+G_b(x,Ty_n,Ty_n)),s(G_b(y_n,x,x)+G_b(x,Ty_n,Ty_n)), \\ G_b(y_n,x,x),G_b(x,Ty_n,Ty_n) \end{pmatrix} \leq 0.$$

According to $(\mathcal{F}_4)$ we have $G_b(x,Ty_n,Ty_n) \leqslant \alpha G_b(x,y_n,y_n)+\beta G_b(y_n,x,x)$ from where $\lim_{n\to\infty} G_b(x,Ty_n,Ty_n)=0$, then (Ty_n) is G_b-convergent to $x=Tx$. Then T is G_b-continuous at x.

In the same way by (1) and (G_{b5}) we find

$$F\begin{pmatrix} G_b(x,Ry_n,Ry_n),G_b(x,y_n,y_n),0,s(G_b(y_n,x,x)+G_b(x,Ry_n,Ry_n)), \\ s(G_b(y_n,x,x)+G_b(x,Ry_n,Ry_n)),G_b(x,Ry_n,Ry_n),G_b(y_n,x,x), \\ s(G_b(y_n,x,x)+G_b(x,Ry_n,Ry_n)),s(G_b(y_n,x,x)+G_b(x,Ry_n,Ry_n)), \\ G_b(y_n,x,x),G_b(x,Ry_n,Ry_n) \end{pmatrix} \leq 0.$$

According to $(\mathcal{F}_4)$ we have $G_b(x,Ry_n,Ry_n) \leqslant \alpha G_b(x,y_n,y_n)+\beta G_b(y_n,x,x)$ from where $\lim_{n\to\infty} G_b(x,Ry_n,Ry_n)=0$, then (Ry_n) is G_b-convergent to $x=Rx$. Then R is G_b-continuous at x.

Corollary 2.2. *Let (X,G_b) be a complete G_b-metric space with coefficient $s \geq 1$, $n,m \in \mathbb{N}$ and $T,R : X \longrightarrow X$ such that $\forall x,y,z \in X$*

$$F\begin{pmatrix} G_b(T^m(x),R^n(y),R^n(z)),G_b(x,y,z),G_b(x,T^m(x),T^m(x)), \\ G_b(y,R^n(y),R^n(y)),G_b(z,R^n(z),R^n(z)),G_b(x,R^n(y),R^n(y)), \\ G_b(y,T^m(x),T^m(x)),G_b(y,R^n(z),R^n(z)),G_b(z,R^n(y),R^n(y)), \\ G_b(z,T^m(x),T^m(x)),G_b(x,R^n(z),R^n(z)) \end{pmatrix} \leq 0,$$

and

$$F\begin{pmatrix} G_b(R^n(x),T^m(y),T^m(z)),G_b(x,y,z),G_b(x,R^n(x),R^n(x)), \\ G_b(y,T^m(y),T^m(y)),G_b(z,T^m(z),T^m(z)),G_b(x,T^m(y),T^m(y)), \\ G_b(y,R^n(x),R^n(x)),G_b(y,T^m(z),T^m(z)),G_b(z,T^m(y),T^m(y)), \\ G_b(z,R^n(x),R^n(x)),G_b(x,T^m(z),T^m(z)) \end{pmatrix} \leq 0,$$

with $F \in \mathcal{F}$. Then, T and R have a unique common fixed point.

Moreover, if F check $(\mathcal{F}_4)$, then T^m and R^n are G_b-continuous at x.

Proof.

Suppose that F check $(\mathcal{F}_4)$, by Theorem 2.2, T^m and R^n have a unique common fixed point $(T^m(x)=R^n(x)=x)$ and T^m,R^n are G_b-continuous at x. Then $Tx=T(T^m(x))=T^{m+1}(x)=T^m(Tx)$, then Tx is also a fixed point for T^m, according to unicity we have $Tx=x$. In the same way we find $Rx=x$.

Then, T and R have a unique common fixed point $(Tx=Rx=x)$ and T^m,R^n are G_b-continuous at x.

From Theorem 2.2 and $T=R$ a we obtain corollary

Corollary 2.3. *Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$, $T{:}X \longrightarrow X$ such that $\forall x, y, z \in X$. :*

$$F\begin{pmatrix} G_b(Tx,Ty,Tz), G_b(x,y,z), G_b(x,Tx,Tx), \\ G_b(y,Ty,Ty), G_b(z,Tz,Tz), G_b(x,Ty,Ty), \\ G_b(y,Tx,Tx), G_b(y,Tz,Tz), G_b(z,Ty,Ty), \\ G_b(z,Tx,Tx), G_b(x,Tz,Tz) \end{pmatrix} \leq 0, \tag{17}$$

with $F \in \mathcal{F}$. Then T has a unique fixed point.

Moreover, if F check $(\mathcal{F}_4)$, then T is G_b-continuous at this point.

3. **Consequences of the main result**

From Corollary 2.3 and Example 2.1 a we obtain [Corollary 4 [3]]

From Corollary 2.3 and Example 2.2 a we obtain [Theorem 6 [3]]

From Corollary 2.3 and Example 2.3 with $s = 1$ a we obtain [Theorem 2.1 [12]].

From Corollary 2.3 and Example 2.5 with $s = 1$ a we obtain [Theorem 31 [5]].

From Corollary 2.3 and Example 2.5, $y = z$ with $s = 1$ a we obtain [Theorem 2.6 [12]].

From Corollary 2.3 and Example 2.6 with $s = 1$ a we obtain [Theorem 2.4 [12]].

From Corollary 2.3 and Example 2.7 with $s = 1$ a we obtain [Theorem 2.8 [12]].

4. **Application**

Let $X = C([a, b], \mathbb{R})$ be the set of real continuous functions defined on $[a, b]$. For $x, y, z \in X$, take the G_b-metric $G_b : X \times X \times X \longrightarrow \mathbb{R}^+$ given by

$$G_b(x,y,z) = (\sup_{t\in[a,b]} \mid x(t) - y(t) \mid + \sup_{t\in[a,b]} \mid x(t) - z(t) \mid + \sup_{t\in[a,b]} \mid y(t) - z(t) \mid)^2. \tag{18}$$

Then (X, G_b) is a complete G_b-metric space with $s = 2$.

Consider the following integral equation

$$x(t) = \int_a^b M(t,u) f(u, x(u)) du, \quad t \in [a,b], \tag{19}$$

where $f : [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function and $M : [a, b] \times [a, b] \longrightarrow \mathbb{R}^+$ is a function such that $M(t, .) \in L^1([a, b])$ for all $t \in [a, b]$.

Consider the operator T: $X \longrightarrow X$ defined by

$$Tx(t) = \int_a^b M(t,u) f(u, x(u)) du, \quad t \in [a,b]. \tag{20}$$

Theorem 4.1. *Suppose that the following conditions are satisfied :*

(H_1) *For all $x(.), y(.) \in X$, $u \in [a, b]$, we have*

$$|f(u,x(u)) - f(u,y(u))| \leq k|x(u) - y(u)| \quad \textit{where} \quad k > 0.$$

(H_2) $\sup_{t\in[a,b]} \int_a^b M(t,u) du \leq \frac{1}{k} e^{-\tau}$, *with* $\tau \in (ln(\sqrt{s}), +\infty)$.

Then the integral equation (19) *has a unique solution in X.*

Proof.

It is clear that any fixed point of (20) is a solution of (19). By conditions (H_1) and (H_2), we have

$$\begin{aligned} G_b(Tx(t),Ty(t),Ty(t)) &= (2\sup_{t\in[a,b]}|Tx(t)-Ty(t)|)^2 \\ &= (2\sup_{t\in[a,b]}|\int_a^b M(t,u)f(u,x(u))du-\int_a^b M(t,u)f(u,y(u))du\,|)^2 \\ &= (2\sup_{t\in[a,b]}|\int_a^b M(t,u)[f(u,x(u))-f(u,y(u))]du\,|)^2 \\ &\le (2k\sup_{t\in[a,b]}\int_a^b M(t,u)\mid x(u)-y(u)\mid du)^2 \\ &\le k^2(2\sup_{t\in[a,b]}\mid x(t)-y(t)\mid)^2(\sup_{t\in[a,b]}\int_a^b M(t,u))du)^2 \\ &\le e^{-2\tau}G_b(x(t),y(t),y(t)). \end{aligned}$$

Then all conditions of Corollary 2.3 are satisfied with $y=z$, and F as in Example 2.2 with $\gamma(t)=e^{-2\tau}$, thus the operator T has a unique fixed point, that is the integral has a unique solution in X.

Example 4.1. *Let $s\ge 1$, the following integral equation has a solution in $X=(C[1,e],\mathbb{R})$.*

$$x(t)=\int_1^e \frac{\ln(ut)}{se}x(u)du,\quad t\in[1,e]. \tag{21}$$

Proof.
Let T: $X\longrightarrow X$ defined by

$$Tx(t)=\int_1^e \frac{\ln(ut)}{se}x(u)du,\quad t\in[1,e].$$

By specifying $M(t,u)=\dfrac{\ln(ut)}{se}$, $f(u,x)=x$ and $\tau\in]ln(\sqrt{s}),ln(s)]$ in Theorem 4.1, we get :
(1) For all $x(.),y(.)\in X$, it is clear that the condition (H_1) in Theorem 4.1 is satisfied with $k=1$.
(2)

$$\begin{aligned} \sup_{t\in[1,e]}\int_1^e \frac{\ln(ut)}{se}du &= \frac{1}{se}\sup_{t\in[1,e]}\int_1^e(\ln(u)+\ln(t))du \\ &= \frac{1}{se}\sup_{t\in[1,e]}[u\ln(u)-u+u\ln(t)]_1^e \\ &= \frac{1}{se}\sup_{t\in[1,e]}(\ln(t)(e-1)+1) \\ &= \frac{1}{s}\le e^{-\tau} \end{aligned}$$

Therefore, all conditions of Theorem 4.1 are satisfied, hence the mapping T has a fixed point in X, which is a solution to equation (21).

5. Acknowledgments.

The authors would like to thank referee sincerely for very helpful comments improving the paper.

The authors declare that there is no conflict of interest regarding the publication of this paper.

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