GENERALIZED DERIVATIONS AND GENERALIZED AMENABILITY OF BANACH ALGEBRAS

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Let \( \mathfrak{A} \) be a Banach algebra. A generalized derivation from \( \mathfrak{A} \) into itself is a linear map \( D \) such that \( D(xa) = D(a)x + xD(a) \) for all \( a, x \in \mathfrak{A} \), where \( d \) is a derivation from \( \mathfrak{A} \) into \( \mathfrak{A} \). In this paper we define dual generalized derivation from Banach algebra \( \mathfrak{A} \) into dual of its \( \mathfrak{A}^* \) or dual of some Banach \( \mathfrak{A} \)-module \( X \) and study its properties.

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1. Introduction

Amenability is a cohomological property of Banach algebras which was introduced by Johnson in [14]. Let \( \mathfrak{A} \) be a Banach algebra, and suppose that \( X \) is a Banach \( \mathfrak{A} \)-bimodule such that the following statements hold
\[
\|a \cdot x\| \leq \|a\|\|x\| \quad \text{and} \quad \|x \cdot a\| \leq \|a\|\|x\|
\]
for each \( a \in \mathfrak{A} \) and \( x \in X \).

We can define the right and left actions of \( \mathfrak{A} \) on dual space \( X^* \) of \( X \) via
\[
\langle x, \lambda \cdot a \rangle = \langle a \cdot x, \lambda \rangle \quad \langle x, a \cdot \lambda \rangle = \langle x \cdot a, \lambda \rangle,
\]
for each \( a \in \mathfrak{A}, x \in X \) and \( \lambda \in X^* \).

Suppose that \( X \) is a Banach \( \mathfrak{A} \)-bimodule. A derivation \( D : \mathfrak{A} \to X \) is a linear map which satisfies \( D(ab) = a \cdot D(b) + D(a) \cdot b \) for each \( a, b \in \mathfrak{A} \) and it is called Jordan derivation in case \( D(x^2) = D(x) \cdot x + x \cdot D(x) \) for each \( x \in \mathfrak{A} \). It is clear that every derivation is a Jordan derivation.

A derivation \( \delta \) is said to be inner if there exists a \( x \in X \) such that \( \delta(a) = \delta_x(a) = a \cdot x - x \cdot a \) for each \( a \in \mathfrak{A} \). We denote the linear space of bounded derivations from \( \mathfrak{A} \) into \( X \) by \( Z^1(\mathfrak{A}, X) \) and the linear subspace of inner derivations by \( N^1(\mathfrak{A}, X) \).

We consider the quotient space \( H^1(\mathfrak{A}, X) = Z^1(\mathfrak{A}, X) / N^1(\mathfrak{A}, X) \), it is called the first Hochschild cohomology group of \( \mathfrak{A} \) with coefficients in \( X \). The Banach algebra \( \mathfrak{A} \) is said to be amenable if \( H^1(\mathfrak{A}, X^*) = \{0\} \) for each Banach \( \mathfrak{A} \)-bimodules \( X \). The Banach algebra \( \mathfrak{A} \) is called weakly amenable if, \( H^1(\mathfrak{A}, X^*) = \{0\} \) (for more details

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A derivation $D : \mathfrak{A} \rightarrow \mathfrak{A}$ is called generalization derivation if there exists a derivation $d : \mathfrak{A} \rightarrow \mathfrak{A}$ such that $D(xy) = D(x)y + xd(y)$ for each pairs $x, y \in \mathfrak{A}$ and we say $D$ is a $d$-derivation. It is easy to see that $D : \mathfrak{A} \rightarrow \mathfrak{A}$ is generalized derivation if and only if $D$ is of the form $D = d + \varphi$, where $d$ is a derivation from $\mathfrak{A}$ into $\mathfrak{A}$ and $\varphi$ is a left module mapping.

The set of bounded $\mathfrak{A}$-module homomorphisms from $\mathfrak{A}$ into an $\mathfrak{A}$-module $M$ is itself an $\mathfrak{A}$-module, when the module operation is given by $a \cdot \phi(x) = \phi(x \cdot a)$ or $\phi(a \cdot x) = \phi(x) \cdot a$, for each $a \in \mathfrak{A}$ and each module homomorphisms $\phi$. This module is denoted by $\text{Hom}(\mathfrak{A}, M)$. A map $T \in \text{Hom}(\mathfrak{A}, \mathfrak{A})$ is called a multiplier, and we write $\text{Hom}(\mathfrak{A}, \mathfrak{A}) = M(\mathfrak{A})$. The set $M(\mathfrak{A})$ is a Banach subalgebra of $B(\mathfrak{A})$, the set of all bounded operators on $\mathfrak{A}$. The homomorphic image of $\mathfrak{A}$ in $M(\mathfrak{A})$ is given by $a \mapsto L_a$, where $L_a(x) = ax$, is called the regular representation of $\mathfrak{A}$.

The generalized derivation $D : \mathfrak{A} \rightarrow \mathfrak{A}$ is inner if there exist $a, b \in \mathfrak{A}$, such that $D(x) = bx - xa$. If we consider $\mathfrak{A}$ as a right $\mathfrak{A}$-module, generalized derivation $\delta : \mathfrak{A} \rightarrow \mathfrak{A}$ is inner if there exist $a \in \mathfrak{A}$ and $\phi \in M(\mathfrak{A})$, such that $\delta(x) = \phi(x) - xa$, that $\phi(x) = bx$.

There are some generalizations for amenability of Banach algebras such as approximate amenability [10], character amenability [15, 17], approximate character amenability [13], ideal amenability [7] and approximate ideal amenability [6]. We denote the linear space of bounded generalized derivations from $\mathfrak{A}$ into $X$ by $GZ^1(\mathfrak{A}, X)$ and the linear subspace of generalized inner derivations by $GN^1(\mathfrak{A}, X)$, we consider the quotient space $GH^1(\mathfrak{A}, X) = GZ^1(\mathfrak{A}, X)/GN^1(\mathfrak{A}, X)$, called the first generalized Hochschild cohomology group of $\mathfrak{A}$ with coefficients in $X$. Similar to amenability of Banach algebra we say $\mathfrak{A}$ is a generalized amenable if $GH(\mathfrak{A}, X^*) = \{0\}$ for every Banach $\mathfrak{A}$-bimodule $X$.

2. Basic Properties

In this section let $\mathfrak{A}$ be a Banach algebra and $M$ be a Banach $\mathfrak{A}$-bimodule. We use \textquotedblleft$\cdot$\textquotedblright for module product between $M$ and its dual and \textquotedblleft$\cdot$\textquotedblright denote the module product between $M$ and $\mathfrak{A}$.

**Definition 2.1.** A linear mapping $\delta : M \rightarrow M^*$ is said to be dual generalized derivation on $M$, if there exist a derivation $d : \mathfrak{A} \rightarrow M^*$ such that

$$\delta(xa) = \delta(x) \cdot a + x.d(a)$$

for each $x \in M$ and for each $a \in \mathfrak{A}$.

**Definition 2.2.** Let $M$ be a Banach algebra and let $\delta : M \rightarrow M^*$ be a dual generalized derivation. $\delta$ is said to be dual generalized inner derivation, if there exist $a, b \in M^*$ such that $\delta(x) = bx - xa$, for each $x \in M$.

As above mentioned, it is proved that $D : \mathfrak{A} \rightarrow \mathfrak{A}$ is generalized derivation if and only if $D$ is of the form $D = d + \varphi$, where $d$ is a derivation from $\mathfrak{A}$ into $\mathfrak{A}$ and
\( \varphi \) is a left module mapping. In the next lemma we extend this for the case when \( D: \mathfrak{A} \rightarrow \mathfrak{A}' \).

**Lemma 2.1.** A linear mapping \( \delta: \mathfrak{A} \rightarrow \mathfrak{A}' \) is dual generalized derivation if and only if there exist a derivation \( d: \mathfrak{A} \rightarrow \mathfrak{A} \) and module map \( \varphi: \mathfrak{A} \rightarrow \mathfrak{A} \) such that \( \delta = d + \varphi \).

**Proof.** Let \( \delta \) be a dual generalized derivation on \( \mathfrak{A} \), so there exist a derivation \( d: \mathfrak{A} \rightarrow \mathfrak{A} \) and module map \( \varphi: \mathfrak{A} \rightarrow \mathfrak{A} \) such that \( \delta = d + \varphi \). Then for each \( a, x \in \mathfrak{A} \), we have

\[
\varphi(xa) = \delta(xa) - d(xa) = \delta(x)a + x.d(a) - (d(x).a + x.d(a)) = \varphi(x)a.
\]

Thus \( \varphi \) is module map and \( \delta = d + \varphi \).

Conversely let \( d \) be a derivation from \( \mathfrak{A} \) to \( \mathfrak{A} \) and \( \varphi: \mathfrak{A} \rightarrow \mathfrak{A} \) be a module map. Take \( \delta = d + \varphi \), then clearly \( \delta \) is a \( d \)-derivation. \( \square \)

**Proposition 2.1.** Let \( \mathfrak{A} \) has a bounded approximate identity and \( \delta: \mathfrak{A} \rightarrow \mathfrak{A} \) be a \( d \)-derivation. Then \( \delta \) is bounded if and only if \( d \) is bounded.

**Proof.** By Lemma 2.1, we can decompose \( \delta \) as \( \delta = d + \varphi \) and by Cohen factorization Theorem [2], \( \varphi \) will be bounded and boundedness of \( \delta \) is only depend on boundedness of \( d \). \( \square \)

**Theorem 2.1.** Let \( \delta: M \rightarrow M^* \) be a bounded linear map. Then \( \delta \) is a dual generalized inner derivation if and only if there exist an inner derivation \( d_a: \mathfrak{A} \rightarrow M^* \) specified by \( a \in \mathfrak{A} \), such that \( \delta \) is a \( d_a \)-derivation.

**Proof.** Let \( \delta \) be a dual generalized derivation. Then there exist \( a, b \in M^* \) such that

\[
\delta(x) = b.x - x.a \quad (x \in M).
\]

Also for every \( x \in M \) we have

\[
\delta(x) \cdot c + x.d_a(c) = (b.x - x.a) \cdot c + xa \cdot c - x \cdot c \cdot a \\
= b.x \cdot c - x.a \cdot c + x.a \cdot c - x \cdot c \cdot a = b.x \cdot c - x \cdot c \cdot a \\
= \delta(x \cdot c).
\]

Thus \( \delta \) is a \( d_a \)-derivation.

Conversely, suppose \( \delta \) is a \( d_a \)-derivation for some \( a \in M^* \). Define \( T: M \rightarrow M^* \) by \( T(x) = \delta(x) + x.a \). Then \( T \) is linear, bounded and for each \( b \in \mathfrak{A} \)

\[
T(x \cdot b) = (\delta(x) + x.a) \cdot b = T(x) \cdot b.
\]

Thus

\[
\delta(x) = (\delta(x) + x.a) - x.a = T(x) - x.a.
\]

Therefore \( \delta \) is a dual generalized inner derivation. \( \square \)

### 3. Main Results

**Proposition 3.1.** Let \( \mathfrak{A}, \mathfrak{B} \) and \( \mathfrak{C} \) be Banach algebras such that \( \mathfrak{A} \) and \( \mathfrak{B} \) are Banach \( \mathfrak{C} \)-bimodule. Suppose that \( \theta: \mathfrak{A} \rightarrow \mathfrak{B} \) is a homeomorphism such that \( \theta \) and \( \theta^{-1} \) are linear module maps and \( d: \mathfrak{C} \rightarrow \mathfrak{C} \) is a derivation. Then for every \( d \)-derivation \( \delta_\mathfrak{B}: \mathfrak{B} \rightarrow \mathfrak{B} \) there exists a \( d \)-derivation \( \delta_\mathfrak{A}: \mathfrak{A} \rightarrow \mathfrak{A} \). Converse is true when \( \theta^{-1} \) is onto.
Proof. Let $\delta_\mathcal{B} : \mathcal{B} \rightarrow \mathcal{B}$ be a $d$-derivation. So for every $y \in \mathcal{B}$ and $c \in \mathcal{C}$ we have
$$\delta_\mathcal{B}(y \cdot c) = \delta_\mathcal{B}(y) \cdot c + y \cdot d(c).$$
Therefore, there exists a $x \in \mathcal{A}$ such that
$$\delta_\mathcal{B}(\theta(x) \cdot c) = \delta_\mathcal{B}(\theta(x)) \cdot c + \theta(x) \cdot d(c).$$
Consequently
$$\delta_\mathcal{B} \circ \theta(x \cdot c) = (\delta_\mathcal{B} \circ \theta(x)) \cdot c + \theta(x) \cdot d(c),$$
and also we have
$$\theta^{-1} \circ \delta_\mathcal{B} \circ \theta(x \cdot c) = (\theta^{-1} \circ \delta_\mathcal{B} \circ \theta(x)) \cdot c + x \cdot d(c) = (\theta^{-1} \circ \delta_\mathcal{B} \circ \theta(x)) \cdot c + x \cdot d(c).$$
Now, assume $\delta_\mathcal{A} = \theta^{-1} \circ \delta_\mathcal{B} \circ \theta$, and so proof is complete.

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\theta$ be defined as above Proposition. Then for every inner $d$-derivation $\delta_\mathcal{B} : \mathcal{B} \rightarrow \mathcal{B}$ there exists a $\delta_\mathcal{A} : \mathcal{A} \rightarrow \mathcal{A}$ such that $\delta_\mathcal{A}$ is an inner $d$-derivation.

**Proposition 3.2.** If $\delta : \mathcal{A} \rightarrow \mathcal{A}$ is a $d$-derivation, then $\delta^{**} : \mathcal{A}^{**} \rightarrow \mathcal{A}^{**}$ is a $d^{**}$-derivation.

**Proof.** It is clear that $\delta^{**}$ is linear. For given $a, b \in \mathcal{A}^{**}$, there exist nets $(a_\alpha)$ and $(b_\beta)$ in $\mathcal{A}$ such that $a = w^* - \lim_\alpha a_\alpha = a$ and $b = w^* - \lim_\beta b_\beta = b$. Then
$$\delta^{**}(ab) = w^* - \lim_\alpha w^* - \lim_\beta \delta(a_\alpha b_\beta) = w^* - \lim_\alpha w^* - \lim_\beta (\delta(a_\alpha)b_\beta + a_\alpha d(b_\beta)) = \delta^{**}(a)b + ad^{**}(b).$$

**Theorem 3.1.** Suppose that the following sequence is a short exact sequence
$$0 \rightarrow \mathcal{J} \xrightarrow{i} \mathcal{A} \xrightarrow{q} \mathcal{B} \rightarrow 0,$$
of Banach algebras, Banach $\mathcal{A}$-bimodules and bounded algebra homomorphism ($\mathcal{A}$ is an extension of $\mathcal{B}$ by $\mathcal{J}$). If $\delta_1 : \mathcal{J} \rightarrow \mathcal{J}^*$ and $\delta_2 : \mathcal{B} \rightarrow \mathcal{B}^*$ be dual generalized $d$-derivations, then there exists a linear map $D : \mathcal{A} \rightarrow \mathcal{A}$ such that $D$ is a dual generalized $d$-derivation.

**Proof.** We may assume that $\mathcal{J}$ is a closed two sided ideal in $\mathcal{A}$ and $\mathcal{B}$ is the quotient space $\mathcal{A}/\mathcal{J}$. According to our assumption we have $\delta_1(xa) = \delta_1(x).a + x.d(a)$ and $\delta_2(ya) = \delta_2(y).a + y.d(a)$, for each $a \in \mathcal{A}$, $x \in \mathcal{J}$ and $y \in \mathcal{A}/\mathcal{J}$.

Now, we define $D = \delta_1 + \delta_2$. It is clear that $D$ is linear and for each $z \in \mathcal{A}$ we have
$$D(za) = D((x + y)a) = D(xa + ya) = D(xa) + D(ya) = \delta_1(xa) + \delta_2(ya) = (\delta_1(x) + \delta_2(y)).a + (x + y)d(a) = D(x + y).a + (x + y).d(a).$$
Thus, $D(za) = D(z).a + z.d(a)$, and proof is complete. □
Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$, be Banach algebras and $D_i : \mathfrak{A}_i \rightarrow \mathfrak{A}_i^*$ be a dual $d_i$-derivation for each $i = 1, \ldots, n$. Then $D : \prod_{i=1}^n \mathfrak{A}_i \rightarrow \prod_{i=1}^n \mathfrak{A}_i^*$ is a dual $d_i$-derivation.

**Proof.** We have the following short exact sequence

$$0 \rightarrow \mathfrak{A}_i \overset{\pi_i}{\longrightarrow} \prod_{i=1}^n \mathfrak{A}_i \overset{\pi_i}{\longrightarrow} \mathfrak{A}_i \rightarrow 0.$$ 

Accordingly to above Theorem, $D$ is a dual $d_i$-derivation. \hfill \Box

**Definition 3.1.** Let $\mathfrak{A}$ be a Banach algebra. We say $\mathfrak{A}$ is generalized amenable if $GH(\mathfrak{A}, X^*) = \{0\}$ for every Banach $\mathfrak{A}$-bimodule $X$.

**Definition 3.2.** Let $\mathfrak{A}$ be a Banach algebra. We say $\mathfrak{A}$ is generalized weakly amenable if $GH(\mathfrak{A}, \mathfrak{A}^*) = \{0\}$.

**Theorem 3.2.** Let $\mathfrak{A}$ be a amenable Banach algebra. Then for every Banach $\mathfrak{A}$-bimodule $M$, we have $GH^1(M, M^*) = \{0\}$.

**Proof.** Let $\delta : M \rightarrow M^*$ be a dual generalized derivation. Then there exists a derivation $d : \mathfrak{A} \rightarrow M^*$ such that $\delta$ is a $d$-derivation. Thus by Theorem 2.1, $GH(M, M^*) = \{0\}$. \hfill \Box

If $\mathfrak{A}$ is an amenable Banach algebra, then for every Banach algebra $M$, which is a Banach $\mathfrak{A}$-bimodule, we have $GH^1(M, M^{(n)}) = \{0\}$ (i.e. $M$ is generalized-n-permanent amenable).

**Theorem 3.3.** Let $\mathfrak{A}$ and $M$ be Banach algebras and $M$ be a right Banach $\mathfrak{A}$-module. If $M$ is weakly amenable, then for every dual generalized $d$-derivation $\delta : M \rightarrow M^*$, $d$ is inner derivation from $\mathfrak{A}$ to $M^*$.

**Proof.** Let $\delta : M \rightarrow M^*$ be a dual generalized $d$-derivation so $\delta(x \cdot b) = \delta(x) \cdot b + x.d(b)$ for $b \in \mathfrak{A}$ and $x \in M$. Since $M$ is weakly amenable, then $\delta$ is an inner derivation. Therefore there exists an $a \in M^*$ such that $\delta(x) = a.x - x.a$.

So we have

$$\delta(x \cdot b) = a.x \cdot b - x \cdot b \cdot a = \delta(x) \cdot b + x.d(b)$$

$$= a.x \cdot b - x.a \cdot b + x.d(b).$$

Then $d(b) = a \cdot b - b \cdot a$ and so $d$ is an inner derivation. \hfill \Box

**Theorem 3.4.** Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$, be Banach algebras and $M_i$ be a Banach $\mathfrak{A}_i$-module for each $i = 1, 2, \ldots, n$. Let $D_i : M_i \rightarrow M_i^*$ be a dual generalized derivation for each $i = 1, \ldots, n$. Then

$$D : \prod_{i=1}^n M_i \rightarrow \prod_{i=1}^n M_i^*$$

is a dual generalized derivation.
Proof. Since each $D_i$ is a dual generalized derivation, therefore there exists a derivation such as $d_i : \mathfrak{A}_i \rightarrow M^*_i$ such that $D_i(a \cdot x) = D_i(a) \cdot x + a.d_i(x)$ for each $a \in M_i$ and $x \in \mathfrak{A}_i$. Define $D : M_1 \times M_2 \times \ldots \times M_n \rightarrow M^*_1 \times M^*_2 \times \ldots \times M^*_n$ by $D = (D_1, \ldots, D_n) = \prod_{i=1}^n D_i$. Then for every $(a_1, a_2, \ldots, a_n) \in \prod_{i=1}^n M_i$ and $(x_1, x_2, \ldots, x_n) \in \prod_{i=1}^n A_i$, we have
\[
D(a \cdot x) = D((a_1, a_2, \ldots, a_n) \cdot (x_1, x_2, \ldots, x_n)) = D((a_1 \cdot x_1, a_2 \cdot x_2, \ldots, a_n \cdot x_n)) \\
= (D_1(a_1, x_1), D_2(a_2, x_2), \ldots, D_n(a_n, x_n)) \\
= (D_1(a_1) \cdot x_1 + a_1.d_1(x_1), \ldots, D_n(a_n) \cdot x_n + a_n.d(x_n)) \\
= (D_1(a_1), \ldots, D_n(a_n)) \cdot (x_1, \ldots, x_n) + (a_1, \ldots, a_n). (d_1(x_1), \ldots, d_n(x_n)).
\]

Now, take $d = (d_1, \ldots, d_2)$. Since each $d_i$ is a derivation, so $d$ is a derivation from $\prod_{i=1}^n \mathfrak{A}_i$ into $\prod_{i=1}^n M^*_i$. Then we have
\[
D(a \cdot x) = D(a) \cdot x + a.d(x),
\]
for every $x \in \mathfrak{A}$ and $a \in M$. Thus, $D$ is a $d$-derivation and proof is complete. \(\square\)

Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$, be generalized amenable Banach algebras, then $\prod_{i=1}^n \mathfrak{A}_i$ is generalized amenable.

4. Results for Triangular Banach Algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebra and suppose that $\mathcal{M}$ is Banach $\mathcal{A}, \mathcal{B}$-module. We define triangular Banach algebra
\[
T = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{M} & \mathcal{B} \end{bmatrix},
\]
with the sum and product being given by the usual $2 \times 2$ matrix operations and internal module actions. The norm on $T$ is
\[
\| \begin{bmatrix} a & m \\ b & \end{bmatrix} \| = \|a\|_\mathcal{A} + \|m\|_\mathcal{M} + \|b\|_\mathcal{B}.
\]

Derivation on triangular Banach algebras have been studied by B. E. Forrest and L. W. Marcoux in [6] and amenability and weak amenability of these algebras are studied in [7] and [11]. $T$ as a Banach space is isomorphic to the $\ell^1$-direct sum of $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$, so we have $T^{(2m-1)} \simeq \mathcal{A}^{(2m-1)} \oplus \mathcal{M}^{(2m-1)} \oplus \mathcal{B}^{(2m-1)}$ and $T^{(2m)} \simeq \mathcal{A}^{(2m)} \oplus \mathcal{M}^{(2m)} \oplus \mathcal{B}^{(2m)}$ for each $m \geq 1$.

When $m = 1$, for every $\tau = \begin{bmatrix} \alpha & \mu \\ \beta & \end{bmatrix} \in T^*$ and $\omega = \begin{bmatrix} x & y \\ z & \end{bmatrix}$, the actions of $\omega$ on $\tau$ and $\tau$ on $\omega$ are given by
\[
\omega \circ \tau = \begin{bmatrix} x \circ \alpha + y \circ \mu & z \circ \mu \\ z \circ \beta & \end{bmatrix} \quad \text{and} \quad \tau \circ \omega = \begin{bmatrix} \alpha \circ x & \mu \circ x \\ \mu \circ y + \beta \circ z & \end{bmatrix}
\]

By above relations and easy calculations we have the following theorem:

**Theorem 4.1.** Let $D : T \rightarrow T^*$ be a bounded dual generalized derivation. Then there exist bounded dual generalized derivations $D_A : \mathcal{A} \rightarrow \mathcal{A}^*$, $D_B : \mathcal{B} \rightarrow \mathcal{B}^*$, and
an element $\gamma_D \in M$ such that
\[
D \begin{bmatrix} x & y \\ z \end{bmatrix} = \begin{bmatrix} D_A(x) - y \circ \gamma_D & \gamma_D \circ x - z \circ \gamma_D \\ D_B(z) + \gamma_D \circ y \end{bmatrix} \quad (x \in A, y \in M, z \in B).
\]

**Theorem 4.2.** Let $\delta_A : A \to A^*$ be a dual generalized derivation. Then $D_{\delta_A} : T \to T^*$ defined by
\[
\begin{bmatrix} x & y \\ z \end{bmatrix} \mapsto \begin{bmatrix} \delta_A(x) & 0 \\ 0 & 0 \end{bmatrix}
\]
is a bounded dual generalized derivation and $\delta_A$ is a dual generalized inner derivation if and only if $D_{\delta_A}$ is a dual generalized inner derivation.

Similarly, for $\delta_B : B \to B^*$ with define $D_{\delta_A} : T \to T^*$ by
\[
\begin{bmatrix} x & y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 \\ \delta_B(z) & 0 \end{bmatrix}
\]
above result is true.

**Proof.** Since $\delta_A$ is a dual generalized derivation thus exist derivation $d : A \to A^*$ such that $\delta_A(xa) = \delta_A(x).a + x.d(a)$, for each $x, a \in A$. Then for every $\omega = \begin{bmatrix} x & y \\ z \end{bmatrix}, \nu = \begin{bmatrix} a & m \\ b & 0 \end{bmatrix} \in T$ we have
\[
D_{\delta_A}(\omega \nu) = D_{\delta_A}\left( \begin{bmatrix} xa & xm + yb \\ zb \end{bmatrix} \right) = \begin{bmatrix} \delta_A(xa) & 0 \\ 0 & 0 \end{bmatrix}
\]
\[
= \begin{bmatrix} \delta_A(x).a & 0 \\ 0 & x.d(a) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ x.d(a) & 0 \end{bmatrix}
\]
\[
= D_{\delta_A}(\omega) \nu + \omega \cdot D_d(\nu),
\]
where $D_d$ is a derivation from $T$ into $T^*$ corrolaryresponding to $d$.

Now suppose that $\delta_A$ is a dual generalized inner derivation, therefore $d$ is a inner and accorrolarying to Lemma 3.3 of [7], $D_d$ is a dual generalized inner derivation and by Theorem 2.1, $D_{\delta_A}$ is a dual generalized inner derivation. Converse by Lemma 3.3 of [7] is clear.

We can write the similar proof for $\delta_B$ and $D_{\delta_B}$, and the above results hold too.

5. Jordan Dual Generalized Derivation

**Definition 5.1.** Let $\mathfrak{A}$ be a Banach algebra. An additive mapping $D : \mathfrak{A} \to \mathfrak{A}$ is generalized Jordan derivation if $D(x^2) = D(x)x + xd(x)$ holds for each $x \in \mathfrak{A}$ where $d : \mathfrak{A} \to \mathfrak{A}$ is a Jordan derivation.

**Definition 5.2.** Let $\mathfrak{A}$ be a Banach algebra. An additive mapping $D : \mathfrak{A} \to \mathfrak{A}^*$ is dual Jordan derivation if $D(x^2) = D(x)x + x.D(x)$ holds for each $x \in \mathfrak{A}$.

**Definition 5.3.** Let $\mathfrak{A}$ be a Banach algebra. An additive mapping $D : \mathfrak{A} \to \mathfrak{A}^*$ is dual generalized Jordan derivation if $D(x^2) = D(x)x + x.d(x)$ holds for each $x \in \mathfrak{A}$ where $d : \mathfrak{A} \to \mathfrak{A}^*$ is a dual Jordan derivation.

**Theorem 5.1.** Let $\mathfrak{A}$ be a semisimple Banach algebra and let $D : \mathfrak{A} \to \mathfrak{A}^*$ be a dual generalized Jordan derivation. Then $D$ is a dual generalized derivation.
Proof. Since $D$ is a dual generalized Jordan derivation, we have

$$D(x^2) = D(x).x + x.d(x) \quad (x \in \mathfrak{A})$$

where $d$ is a dual Jordan derivation from $\mathfrak{A}$ into $\mathfrak{A}^*$. Since $\mathfrak{A}$ is a semisimple, then $d$ is a derivation. Define $\varphi = D - d$, then we have

$$\varphi(x^2) = D(x^2) - d(x^2) = D(x).x + x.d(x) - (x.d(x) + d(x).x)$$

$$= D(x).x - d(x).x = (D(x) - d(x)).x = \varphi(x).x$$

therefore $\varphi(x^2) = \varphi(x).x$, for each $x \in \mathfrak{A}$. By Proposition 1.4 of [14], we conclude that $\varphi$ is a module map. Hence $D = \varphi + d$, where $\varphi$ is a module map and $d$ is a derivation. Then by Lemma 2.1, $D$ is a dual generalized derivation. \qed

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