PSEUDOINVEX FUNCTIONS ON RIEMANNIAN MANIFOLDS AND APPLICATIONS IN FRACTIONAL PROGRAMMING PROBLEMS

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In this paper, we have obtained pseudoinvex functions from the ratio of invex and related functions to an affine and some generalized invex functions on Riemannian manifolds. Further, we establish sufficient optimality conditions and duality theorems for fractional nonlinear optimization problems under weaker assumptions on Riemannian manifolds.

Keywords: Riemannian manifolds, fractional programming problems, pseudoinvex functions, optimality conditions, duality

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1. **Introduction**

The ratio of a convex function to a positive affine function is a pseudoconvex function, see, Cambini and Martein [2]. The concept of invex and generalized invex functions was given by Hanson [8] and the same on Riemannian manifolds was given by Pini [14] and Barani and Pouryayevali [1]. Now, we extend the results of Cambini and Martein [2] to Riemannian manifolds. In our case the ratio of an invex function to a positive affine function is a pseudoinvex function and some other similar results also.

A nonlinear fractional programming problem is an optimization problem. In the applications of fractional programming the quotient of two functions is to be maximized or minimized. If f is convex, g is concave and h is convex then the fractional programming is defined as convex-concave fractional programming problem. If all the functions f, g and h are invex then the programming is known as an invex problem. For invex fractional programming problems, we may cited the fundamental work of Craven [4] in which the invex function was advised first time and also the work of Reddy and Mukherjee [17], Singh and Hanson [18], Mishra and Giorgio [13], Craven [3] and Craven and Glover [5]. We continue this fractional programming to pseudoinvex case.

On the other hand, a manifold is not a linear space and extensions of results and techniques from linear spaces to Riemannian manifolds are natural. The importance of the extension is that, with the significant Riemannian metric the nonconvex optimization problems become convex optimization problems. In recent years, many important results and techniques have been developed on various aspects of convex optimization on Riemannian manifolds, see, [19]. We extend the results of Craven and Mond [6] on Riemannian theory and in our case f is invex, g is positive and affine then the objective function is pseudoinvex. Rapcsák [16] proposed optimality conditions for an optimization problem with constraints on smooth manifold. Further, Jana and Nahak [9] have established sufficient optimality conditions and duality results for an optimization problem on a differentiable manifold.

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Udriste [19] introduced the duality results on Riemannian manifolds for a convex programming problem. Further, Ferrara and Mititelu [7] established the Mond-Weir duality results on a smooth manifold for vector optimization problems.

Motivated by the work of Khan and Hanson [10], Udriste [19] and Jana and Nahak [9] we extend the results of Khan and Hanson [10] under assumptions of pseudoinvexity of objective function and of quasiinvexity of constraints on Riemannian manifolds.

This paper is organized as follows : In the next section, we present some preliminaries definitions and concepts of Riemannian manifolds. In section 3, we consider the nonlinear fractional programming problems and obtain pseudoinvex functions from the ratio of invex and related functions to an affine and some generalized invex functions on Riemannian manifolds. Further, we establish sufficient optimality conditions in the presence of pseudoinvex functions. Finally, in the last section, we establish duality results for a fractional nonlinear programming problem on Riemannian manifolds.

2. **Preliminaries**

In this section, we remember some known definitions and concepts of Riemannian manifolds which will be useful for this paper. For standard material of differential geometry, see, Klingenberg [11], Lang [12], and Prakash [15].

Definition 2.1. [9] A curve on a differentiable manifold M is a smooth map γ from some interval $(-\epsilon, \epsilon)$ of the real line to M.

Definition 2.2. [9] A tangent vector on a curve γ at a point y of M is

$$\dot{\gamma_y} : C^{\infty}(M) \to \mathbb{R}, \quad f \longmapsto \dot{\gamma_y}(f) \equiv \frac{d}{dt}(fo\gamma)|_y$$

Definition 2.3. [9] The set of all tangent vectors at point $y \in M$ is defined as the tangent space at y denoted by T_yM.

Definition 2.4. [7] Let M be a differentiable manifold. Let T_yM denote the tangent space to M at y. Then

$$TM = \bigcup_{y \in M} T_yM$$

be the tangent bundle of M.

Let M be a smooth manifold modelled on a Hilbert space H, either finite or infinite dimensional, endowed with a Riemannian metric $f_y(.,.)$ on the tangent space $T_yM \cong H$. Then, we have a smooth inner product to each tangent space, written as

$$f_y(u, v) = \langle u, v\rangle_y, \quad for\ all\ u, v \in T_yM$$

Therefore, M is now a Riemmanian manifold.

Definition 2.5. [9] A manifold whose tangent spaces are endowed with a smooth varying inner product with respect to a point y of M is called a Riemannian manifold. The smoothly varying inner product, denoted as $\langle \xi_y, \zeta_y\rangle$ for every two elements of T_yM, is called a Riemannian metric. If M is a smooth manifold, then there exist always Riemannian metrices on M.

Definition 2.6. [7] Let $f : M \to \mathbb{R}$ be a differentiable function. The differential of f at y, namely $df_y : T_yM \to \mathbb{R}$, is given by

$$df_y(v) = df(y)v, \quad v \in T_yM,$$

and for the Riemannian manifold $(M, \langle ., .\rangle)$ by

$$df_y(v) = \langle df(y), v\rangle_y, \qquad v \in T_yM,$$

where $\langle ., .\rangle$ is the Riemannian metric.

Throughout this paper, let M be a Riemannian manifold and $f : M \to \mathbb{R}$ be a differentiable function and $\eta : M \times M \to TM$ be a function such that for every $x, y \in M$, $\eta(x, y) \in T_yM$.

The function f is said to be invex with respect to η on M, if for all $x, y \in M$, we have

$$f(x) - f(y) \geq df_y(\eta(x, y)).$$

The function f is said to be incave with respect to η on M, if for all $x, y \in M$, we have

$$f(x) - f(y) \leq df_y(\eta(x, y)).$$

Moreover, the function f is affine with respect to η on M, if for all $x, y \in M$, we have

$$f(x) - f(y) = df_y(\eta(x, y)).$$

Further, we introduce the following classes of generalized invex functions :

The function f is said to be pseudoinvex with respect to η on M, if for all $x, y \in M$, we have

$$df_y(\eta(x, y)) \geq 0 \Longrightarrow f(x) \geq f(y).$$

Again f is said to be strictly pseudoinvex with respect to η on M, if for all $x, y \in M$ with $x \neq y$, we have

$$df_y(\eta(x, y)) \geq 0 \Longrightarrow f(x) > f(y).$$

The function f is said to be quasiinvex with respect to η on M, if for all $x, y \in M$, we have

$$f(x) \leq f(y) \Longrightarrow df_y(\eta(x, y)) \leq 0.$$

3. **Sufficient Optimality**

Consider the nonlinear fractional programming problem on a Riemannian manifold M :

(FP) min $\frac{f(x)}{g(x)}$
subject to $h(x) \leq 0,\ x \in M$,

where f,g and h all are defined and differentiable functions on M.
$f, g : M \to \mathbb{R}$ be real-valued functions and $h : M \to \mathbb{R}^m$ be an m-dimensional vector-valued function.

We need the following interesting results for sufficient optimality conditions.

Theorem 3.1. *Consider the ratio $z(x) = \frac{f(x)}{g(x)}$, where f and g are differentiable functions defined on an open Riemannian manifold M.*
(i) If f is invex, and g is positive and affine, then z is pseudoinvex;
(ii) If f is non-negative and invex, and g is positive and incave, then z is pseudoinvex;
(iii) If f is positive and strictly invex, and g is positive and incave, then z is strictly pseudoinvex;
(iv) If f is non-negative and invex, and g is positive and strictly incave, then z is strictly pseudoinvex;
(v) If f is negative and strictly invex, g is positive and invex, then z is strictly pseudoinvex;
(vi) If f is negative and invex, g is positive and strictly invex, then z is strictly pseudoinvex.

Proof. (i) Since f is invex, then for all $x, y \in M$, we have

$$f(x) - f(y) \geq df_y(\eta(x, y)). \tag{1}$$

Again g is affine, then for all $x, y \in M$, we have

$$g(x) - g(y) = dg_y(\eta(x, y)). \tag{2}$$

To show $z = \frac{f}{g}$ is pseudoinvex.

Suppose,

$$d\left(\frac{f}{g}\right)_y(\eta(x,y)) \geq 0,\ for\ all\ x,y \in M.$$

$$\left[\frac{g(y)df_y - f(y)dg_y}{(g(y))^2}\right](\eta(x,y)) \geq 0,$$

$$\left[\frac{df_y}{g(y)} - \frac{f(y)dg_y}{(g(y))^2}\right](\eta(x,y)) \geq 0,$$

$$\frac{1}{g(y)}df_y(\eta(x,y)) - \frac{f(y)}{(g(y))^2}dg_y(\eta(x,y)) \geq 0.$$

Since g is positive, therefore

$$df_y(\eta(x,y)) \geq \frac{f(y)}{g(y)}dg_y(\eta(x,y)),$$

$$f(x) - f(y) \geq df_y(\eta(x,y)) \geq \frac{f(y)}{g(y)}(g(x) - g(y)),\ (by\ (1)\ and\ (2))$$

$$f(x) - f(y) \geq \frac{f(y)}{g(y)}(g(x) - g(y)),$$

$$f(x)g(y) - f(y)g(y) \geq f(y)g(x) - f(y)g(y),$$

$$f(x)g(y) \geq f(y)g(x),$$

$$\frac{f(x)}{g(x)} \geq \frac{f(y)}{g(y)}.$$

Therefore, $\frac{f}{g}$ is pseudoinvex on M.

(v) Since f is negative, let $f = -k$.

Again f is strictly invex, then for all $x, y \in M$, we have

$$f(x) - f(y) > df_y(\eta(x,y)),$$

$$-k(x) + k(y) > d(-k)_y(\eta(x,y)),$$

$$-k(x) + k(y) > -dk_y(\eta(x,y)),$$

$$k(x) - k(y) < dk_y(\eta(x,y)). \tag{3}$$

Now g is invex, then for all $x, y \in M$, we have

$$g(x) - g(y) \geq dg_y(\eta(x,y)). \tag{4}$$

To show $z = \frac{f}{g}$ is strictly pseudoinvex.

Suppose,

$$d\left(\frac{f}{g}\right)_y(\eta(x,y)) \geq 0,\ for\ all\ x,y \in M\ with\ x \neq y.$$

$$d\left(\frac{-k}{g}\right)_y(\eta(x,y)) \geq 0,$$

$$\left[\frac{-g(y)dk_y + k(y)dg_y}{(g(y))^2}\right](\eta(x,y)) \geq 0,$$

$$\left[\frac{-dk_y}{g(y)} + \frac{k(y)dg_y}{(g(y))^2}\right](\eta(x,y)) \geq 0,$$

$$-\frac{1}{g(y)}dk_y(\eta(x,y)) + \frac{k(y)}{(g(y))^2}dg_y(\eta(x,y)) \geq 0.$$

Since g is positive, therefore

$$\frac{k(y)}{g(y)} dg_y(\eta(x,y)) \geq dk_y(\eta(x,y)),$$

$$\frac{k(y)}{g(y)}(g(x)-g(y)) \geq \frac{k(y)}{g(y)} dg_y(\eta(x,y)) \geq dk_y(\eta(x,y)) > (k(x)-k(y)),\ (by\ (3)\ and\ (4))$$

$$\frac{k(y)}{g(y)}(g(x)-g(y)) > (k(x)-k(y)),$$

$$k(y)g(x) - k(y)g(y) > k(x)g(y) - k(y)g(y),$$

$$k(y)g(x) > k(x)g(y).$$

Put $k = -f$ in above we get,

$$-f(y)g(x) > -f(x)g(y),$$

$$f(x)g(y) > f(y)g(x),$$

$$\frac{f(x)}{g(x)} > \frac{f(y)}{g(y)}.$$

Therefore, $\frac{f}{g}$ is strictly pseudoinvex on M. □

Remaining parts can be proved on the lines of similar arguments.

Remark 3.1. A particular case of the above results given by Cambini and Martein [2] is as follows :

Theorem 3.2. *Consider the ratio $z(x) = \frac{f(x)}{g(x)}$, where f and g are differentiable functions defined on an open convex set $X \subseteq \mathbb{R}^n$.*
(i) If f is convex, and g is positive and affine, then z is pseudoconvex;
(ii) If f is non-negative and convex, and g is positive and concave, then z is pseudoconvex;
(iii) If f is positive and strictly convex, and g is positive and concave, then z is strictly pseudoconvex;
(iv) If f is non-negative and convex, and g is positive and strictly concave, then z is strictly pseudoconvex;
(v) If f is negative and strictly convex, g is positive and convex, then z is strictly pseudoconvex;
(vi) If f is negative and convex, g is positive and strictly convex, then z is strictly pseudoconvex.

Example 3.1. Define $z(x) : M \to \mathbb{R}$, by

$$z(x) = \frac{x^2+x+4}{x+5}, \qquad x+5>0,\ x \in M.$$

To show z is pseudoinvex with respect to $\eta(x,y) = x - y$.

Let $f(x) = x^2 + x + 4$. Then for all $x, y \in M$, we have

$$\begin{aligned} f(x) - f(y) - \langle df_y, \eta(x,y)\rangle_y &= (x^2-y^2) + (x-y) - \langle 2y+1, x-y\rangle \\ &= (x-y)(x+y+1-2y-1) \\ &= (x-y)^2 \geq 0, \end{aligned}$$

which is always true.
Therefore,
$f(x) - f(y) - \langle df_y, \eta(x,y)\rangle_y \geq 0$.
Therefore, $f(x)$ is invex with respect to $\eta(x,y)$ on M.
Let $g(x) = x + 5, \qquad x + 5 > 0$.
Since $g(x)$ is composition of a linear function and a constant so $g(x)$ is an affine function.

Now $g(x)$ is positive and affine function with respect to $\eta(x,y)$ on M.
Therefore, $z = \frac{f}{g}$ is pseudoinvex function with respect to $\eta(x,y) = x - y$ on M.

The following constraint qualification [9] will be needed in the sequel :
Let D be the set of all feasible solutions of (FP). Let $\bar{x} \in D$ be an optimal solution of (FP) and we define the set

$$J^0 = \{j \in 1, ..., m : h_j(\bar{x}) = 0\}.$$

Suppose that the domain D satisfies the following constraint qualification at $\bar{x}$:

$$R(\bar{x}) : \exists v \in TM : d(h_{J^0})_{\bar{x}}(v) \leq 0,$$

where $d(h_{J^0})_{\bar{x}}(v)$ is the vector components of $d(h_j)_{\bar{x}}(v)$, $\forall j \in J^0$, taken in increasing order of j.

Remark 3.2. If a feasible point $x^0 \in M$ be an optimal solution of the problem (FP) and satisfies the constraint qualification $R(\bar{x})$, then the following Kuhn-Tucker conditions are necessary for (FP) :

$$d\left(\frac{f}{g}\right)_{x^0}(\eta(x,x^0)) + \lambda^{0^T} dh_{x^0}(\eta(x,x^0)) = 0, \tag{5}$$

$$\lambda^{0^T} h(x^0) = 0, \tag{6}$$

$$\lambda^0 \geq 0. \tag{7}$$

Further, the next theorem proves that if the functions in (FP) are under suitable invexity, then the conditions (5)-(7) are sufficient for optimality.

Theorem 3.3. *Suppose that $x^0 \in M$ be feasible for (FP), and that the Kuhn-Tucker conditions (5)-(7) are satisfied at x^0. Let f be invex, g be positive and affine and h be quasiinvex with respect to η on M. Then x^0 is a minimum for (FP).*

Proof. Let x^0 be a feasible point for problem (FP).
Since f be invex, g be positive and affine with respect to η on M, then by Theorem (3.1), $\frac{f}{g}$ be pseudoinvex with respect to η on M.
Since h be quasiinvex with respect to η on M, then for all $x, x^0 \in M$, we have

$$h(x) - h(x^0) \leq 0 \Longrightarrow dh_{x^0}(\eta(x,x^0)) \leq 0,$$

$$\Longrightarrow \lambda^{0^T} dh_{x^0}(\eta(x,x^0)) \leq 0, \qquad \text{(since } \lambda^0 \geq 0)$$

$$\Longrightarrow -d\left(\tfrac{f}{g}\right)_{x^0}(\eta(x,x^0)) \leq 0, \qquad \text{(by (5))}$$

$$\Longrightarrow d\left(\tfrac{f}{g}\right)_{x^0}(\eta(x,x^0)) \geq 0.$$

Using pseudoinvexity of $\frac{f}{g}$ with respect to η on M, we have

$$\frac{f(x)}{g(x)} \geq \frac{f(x^0)}{g(x^0)}.$$

Therefore, x^0 is a global minimum. □

Remark 3.3. It should be observed that Khan and Hanson [10] noticed that the fractional programming problem is an invex problem but under weaker assumptions our above results show that (FP) is also a pseudoinvex problem on Riemannian manifolds under appropriate assumptions of pseudoinvexity of the objective function and of quasiinvexity of the constraints. Therefore our result is stronger than that of Khan and Hanson [10].

4. **Duality Results**

Consider the following Mond-Weir dual model for (FP) :

$$\text{(FD)}\ max\ \tfrac{f(v)}{g(v)}$$

$$subject\ to\ \ d\left(\frac{f}{g}\right)_v(\eta(x^0,v)) + \lambda^T dh_v(\eta(x^0,v)) = 0, \tag{8}$$

$$\lambda^T h(v) = 0, \tag{9}$$

$$\lambda \geq 0. \tag{10}$$

Theorem 4.1. ***(Weak duality)*** *Let x^0 be feasible for primal problem (FP) and v be feasible for dual problem (FD) then*

$$\frac{f(x^0)}{g(x^0)} \geq \frac{f(v)}{g(v)}.$$

Proof. Since x^0 be feasible for (FP) and (v,λ) be feasible for (FD), then the above Kuhn-Tucker conditions (8)-(10) hold.
Since h be quasiinvex with respect to η on M, then for all $x^0, v \in M$, we have

$$\begin{aligned} h(x^0) - h(v) \leq 0 &\Longrightarrow dh_v(\eta(x^0,v)) \leq 0, \\ &\Longrightarrow \lambda^T dh_v(\eta(x^0,v)) \leq 0, \qquad (\text{since } \lambda \geq 0) \\ &\Longrightarrow -d\big(\tfrac{f}{g}\big)_v(\eta(x^0,v)) \leq 0, \qquad (\text{by (8)}) \\ &\Longrightarrow d\big(\tfrac{f}{g}\big)_v(\eta(x^0,v)) \geq 0. \end{aligned}$$

By the pseudoinvexity of $\frac{f}{g}$ with respect to η on M, we have

$$\frac{f(x^0)}{g(x^0)} \geq \frac{f(v)}{g(v)}.$$

□

Remark 4.1. In the following, suppose the functions f be invex, g be positive and affine with respect to $\eta(x^0,v)$ on M, then $\frac{f}{g}$ be pseudoinvex (according to Theorem 3.1) with respect to $\eta(x^0,v)$ on M, so $\lambda^T h(.); \lambda \geq 0$ is pseudoinvex with respect to $\eta(x^0,v)$ on M.

Theorem 4.2. ***(Strong duality)*** *Under the Kuhn-Tucker conditions, suppose x^0 be minimal for (FP) then there exists $0 \leq \lambda^0 \in \mathbb{R}^n$ such that (x^0,λ^0) be maximal for (FD) and the optimal values of (FP) and (FD) are equal.*

Proof. Let any vector (v,λ) also satisfies the constraints of (FD), then (v,λ) satisfies the Kuhn-Tucker conditions as follows :

$$d\left(\frac{f}{g}\right)_v(\eta(x^0,v)) + \lambda^T dh_v(\eta(x^0,v)) = 0, \tag{11}$$

$$\lambda^T h(v) = 0, \tag{12}$$

$$\lambda \geq 0. \tag{13}$$

To prove (x^0,λ^0) is maximal for (FD), we have to show

$$\frac{f(x^0)}{g(x^0)} - \frac{f(v)}{g(v)} \geq 0.$$

Since h be quasiinvex with respect to $\eta(x^0,v)$ on M, then we have

$$h(x^0) - h(v) \leq 0 \Longrightarrow dh_v(\eta(x^0,v)) \leq 0,$$

$$\implies \lambda^T dh_v(\eta(x^0, v)) \leq 0, \qquad \text{(by (13))}$$

$$\implies -\lambda^T dh_v(\eta(x^0, v)) \geq 0. \tag{14}$$

From the constraint (12), we have

$$\begin{aligned} d\left(\tfrac{f}{g}\right)_v(\eta(x^0, v)) &\geq d\left(\tfrac{f}{g}\right)_v(\eta(x^0, v)) - \lambda^T h(v), \\ &\geq -\lambda^T dh_v(\eta(x^0, v)) - \lambda^T h(v), \qquad \text{(from (11))} \\ &\geq -\lambda^T h(v), \qquad \text{(from (14))} \end{aligned}$$

$$\geq -\lambda^T h(x^0),$$

(since $\lambda^T h(.)$ is pseudoinvex with respect to $\eta(x^0, v)$ on M)

Therefore,

$$d\left(\frac{f}{g}\right)_v(\eta(x^0, v)) \geq 0,$$

since $h(x^0) \leq 0$ and by using (13).

By the pseudoinvexity of $\frac{f}{g}$ with respect to $\eta(x^0, v)$ on M, we have

$$\frac{f(x^0)}{g(x^0)} - \frac{f(v)}{g(v)} \geq 0.$$

Therefore, (x^0, λ^0) is maximal for (FD) and objective values are equal in both problems. □

Theorem 4.3. ***(Converse duality)*** *If (x^0, λ^0) be maximal for (FD) and a dual constraint qualification $R(\bar{x})$ be satisfied at (x^0, λ^0), then x^0 be minimal for (FP).*

Proof. Since a constraint qualification $R(\bar{x})$ be satisfied at x^0, then the following Kuhn-Tucker conditions hold at (x^0, λ^0), i.e.

$$d\left(\frac{f}{g}\right)_{x^0}(\eta(x, x^0)) + \lambda^{0^T} dh_{x^0}(\eta(x, x^0)) = 0, \tag{15}$$

$$\lambda^{0^T} h(x^0) = 0, \tag{16}$$

$$\lambda^0 \geq 0. \tag{17}$$

Since h be quasiinvex with respect to $\eta(x, x^0)$ on M, therefore

$$\begin{aligned} h(x) - h(x^0) \leq 0 &\implies dh_{x^0}(\eta(x, x^0)) \leq 0, \\ &\implies \lambda^{0^T} dh_{x^0}(\eta(x, x^0)) \leq 0, \qquad \text{(by (17))} \end{aligned}$$

$$\implies -\lambda^{0^T} dh_{x^0}(\eta(x, x^0)) \geq 0. \tag{18}$$

Since $\lambda^{0^T} h(.)$ be pseudoinvex with respect to $\eta(x, x^0)$ on M, therefore

$$\begin{aligned} \lambda^{0^T} dh_{x^0}(\eta(x, x^0)) \geq 0 &\implies \lambda^{0^T} h(x) \geq \lambda^{0^T} h(x^0), \\ &\implies \lambda^{0^T} h(x^0) - \lambda^{0^T} h(x) \leq 0. \end{aligned} \tag{19}$$

For any $x \in M$, satisfying the constraints of (FP), we have

$$\begin{aligned} d\left(\frac{f}{g}\right)_{x^0}(\eta(x, x^0)) &= -\lambda^{0^T} dh_{x^0}(\eta(x, x^0)), \qquad \textit{(by (15))} \\ &\geq -\lambda^{0^T} dh_{x^0}(\eta(x, x^0)) + \lambda^{0^T} h(x^0) - \lambda^{0^T} h(x), \end{aligned}$$

$$\text{(using (19))}$$
$$\geq -\lambda^{0^T} h(x), \qquad (by\ (16)\ and\ (18)).$$

Therefore,

$$d\left(\frac{f}{g}\right)_{x^0}(\eta(x,x^0)) \geq 0, \qquad being\ h(x) \leq 0\ and\ by\ (17).$$

Since $\frac{f}{g}$ be pseudoinvex with respect to $\eta(x,x^0)$ on M, then we have

$$\frac{f(x)}{g(x)} - \frac{f(x^0)}{g(x^0)} \geq 0.$$

Hence, x^0 is minimum for (FP). □

Remark 4.2. Khan and Hanson [10] gave similar results assuming that f is invex, g is positive and incave but under weaker assumptions on Riemannian manifolds that f is invex, g is positive and affine our objective function is a generalization of objective function of Khan and Hanson [10]. Therefore our duality results are more general than that of Khan and Hanson [10].

5. Conclusions

We obtained pseudoinvex functions from the ratio of invex and related functions to an affine and some other generalized invex functions on Riemannian manifolds. Again, we consider the nonlinear fractional programming problems and established the sufficient optimality conditions and duality theorems under appropriate assumptions of pseudoinvexity of the objective function and of quasiinvexity of the constraints on Riemannian manifolds. In this way under weaker assumptions our optimality and duality results are more general.

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