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CHARACTERIZING SUBDIFFERENTIAL OF DECREASING INVERSE CO-RADIANT FUNCTIONS WITH APPLICATIONS

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In this paper, we first characterize vector valued decreasing inverse co-radiant functions in a framework of abstract convexity. Next, we show that each vector valued decreasing inverse co-radiant function is supremally generated by a certain class of elementary functions. Finally, as an application, the basic properties of this class of functions such as support set and subdifferential are characterized.

Keywords: decreasing function, inverse co-radiant function, subdifferential, support set, elementary function, abstract convexity, conditionally complete Banach lattice.

MSC2010: 26A 48; 26A 27; 26A 09; 26B 25; 06B 23.

1. Introduction

Abstract convexity has found many applications in mathematical analysis and optimization problems [8, 14, 15]. One of the main questions, that arises in abstract convexity, is to identify a small supremal generator of sets of abstract convex functions. In general, the identification of such a generator is not a simple task. Indeed, the fact that the set of affine functions is a supremal generator of the set of lower semi-continuous convex functions is equivalent to the Hahn-Banach theorem. So, it is beneficial to have a small enough supremal generator, which consists of simple functions. It is well-known that some classes of increasing functions are abstract convex. For example, the class of increasing and positively homogeneous (IPH) functions [5, 6, 11] and the class of increasing and convex-along-rays functions are abstract convex [12, 13]. The class of increasing and co-radiant functions is another class of increasing functions, which is abstract convex [3, 4]. In a forthcoming study, our main goal is to investigate the optimization of dual functions of increasing co-radiant

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and quasi-concave functions. In [9], it has been shown that the dual function of an increasing co-radiant and quasi-concave function is decreasing inverse co-radiant. This is a motivation that we first study the class of decreasing inverse co-radiant functions. In this paper, we first characterize vector valued decreasing inverse co-radiant functions with respect to a certain class of elementary functions. Next, we show that this class of functions is abstract convex. Finally, as an application, we present characterizations of support set and subdifferential for this class of functions.

The structure of the paper is as follows: In Section 2, we provide some preliminaries and definitions relative to abstract convexity and Banach lattices. In Section 3, we characterize vector valued decreasing inverse co-radiant functions with respect to a certain class of elementary functions. Next, we show that the class of vector valued decreasing inverse co-radiant functions is abstract convex. Finally, as an application, characterizations of support set and subdifferential of this class of functions are given in Section 4. Section 5, includes with a discussion on conclusions and applications.

2. **Preliminaries**

Let X be a real topological vector space. We assume that X is equipped with a closed convex pointed cone S (the latter means that $S \cap (-S) = \{0\}$). Assume $S \neq \{0\}$. The increasing property of our functions will be understood to be with respect to the ordering $\leq$ induced on X by S :

$$x \leq y \quad \Leftrightarrow \quad y - x \in S, \ (x, \ y \in X).$$

We say $x < y$ if and only if $y - x \in S \setminus \{0\}$. Throughout the paper we assume that X is not a finite set.

In the following, we recall some notions and definitions concerning vector lattices (see [1, 2, 7, 10]).

Definition 2.1. *A partially ordered set Y is a set in which a binary relation $\preceq$ is defined, which satisfies the following conditions:*
(1) *For all $x \in Y$, $x \preceq x$ (reflexivity).*
(2) *If $x, y \in Y$ are such that $x \preceq y$ and $y \preceq x$, then $x = y$ (antisymmetry).*
(3) *If $x, y, z \in Y$ are such that $x \preceq y$ and $y \preceq z$, then $x \preceq z$ (transitivity).*

If $x, y \in Y$ are such that $x \preceq y$ and $x \neq y$, one writes $x \prec y$, and says that x "is less than "y. The relation $x \preceq y$ is also written $y \succeq x$, and similarly, $x \prec y$ is also written $y \succ x$.

Definition 2.2. *A partially ordered set $(Y, \preceq)$ is a lattice if each pair of elements $x, y \in Y$ has a least upper bound (a supremum) and a greatest lower bound (an infimum). The supremum and infimum of any two elements $x, y \in Y$ is denoted by*

$$\sup\{x, y\} := x \vee y \ \textit{and} \ \inf\{x, y\} := x \wedge y,$$

where the mappings

$$\begin{aligned} \vee : Y \times Y &\longrightarrow Y \\ (x,y) &\longmapsto x \vee y \end{aligned}$$

and

$$\begin{aligned} \wedge : Y \times Y &\rightarrow Y \\ (x,y) &\longmapsto x \wedge y \end{aligned}$$

are called the lattice operations on Y*.*

Note that both the supremum and infimum are unique, provided that they exist.

If, in addition, Y is a real vector space, then, Y is called an ordered vector space if, for all $x, y \in Y$, the following assertions hold:
(1) If $x \preceq y$, then, $x + z \preceq y + z$ for all $z \in Y$.
(2) If $x \preceq y$, then, $\alpha x \preceq \alpha y$ for all $\alpha \geq 0$.

Definition 2.3. *A lattice* $(L, \preceq)$ *is said to be conditionally complete if it satisfies one of the following equivalent conditions:*
(1) *Every non-empty lower bounded set admits an infimum.*
(2) *Every non-empty upper bounded set admits an supremum.*
(3) *There exists a complete lattice* $\overline{L} := L \cup \{\perp, \top\}$*, which we call the minimal completion of* L*, with bottom element* $\perp$ *and top element* $\top$ *such that* L *is a sublattice of* $\overline{L}$*,* $\inf L := \perp$ *and* $\sup L := \top$*.*

Definition 2.4. *A (real) vector lattice* $(Y, \preceq, +, \cdot)$ *is a set* Y *endowed with a partial order* $\preceq$ *such that* $(Y, \preceq)$ *is a lattice with a binary operation "* $+$ *" and the scalar product "* $\cdot$ *" such that* $(Y, +, \cdot)$ *is a vector space.*

Definition 2.5. *A vector lattice* $(Y, \preceq, +, \cdot)$ *such that* $(Y, \preceq)$ *is a conditionally complete lattice is called a conditionally complete vector lattice.*

Definition 2.6. *A conditionally complete Banach lattice (resp. normed lattice)* Y *is a (real) Banach space (resp. normed space) which is also a conditionally complete lattice such that*

$$|x| \preceq |y| \Longrightarrow \|x\| \leq \|y\|, \quad \forall\, x,\, y \in Y, \tag{1}$$

where $|x| := x^+ + x^-$*,* $x^+ := \sup\{x, 0\}$ *and* $x^- := \sup\{-x, 0\}$*.*

Let Y be a vector lattice. Recall (see [2, 7, 10]) that an element $\mathbf{1} \in Y$ is called a strong unit, if for each $x \in Y$ there exists $0 < \lambda \in \mathbb{R}$ such that $x \preceq \lambda \mathbf{1}$. Now, consider the vector lattice Y with the strong unit $\mathbf{1}$. By using the strong unit $\mathbf{1}$, we can define a norm on Y by

$$\|x\| := \inf\{\lambda > 0 : |x| \preceq \lambda \mathbf{1}\}, \quad \forall\, x \in Y. \tag{2}$$

It is easy to check that $\|.\|$ is a norm on Y, and satisfies

$$|x| \preceq \|x\| \mathbf{1}, \quad \forall\, x \in Y. \tag{3}$$

Then, in view of (2) and (3), we conclude that

$$\begin{aligned} B(x,r) &:= \{y \in Y : \|y - x\| \leq r\} \\ &= \{y \in Y : x - r\mathbf{1} \preceq y \preceq x + r\mathbf{1}\}, \end{aligned}$$

where $0 < r \in \mathbb{R}$ and $x \in Y$. It is clear that the ball $B(x, r)$ is a closed and convex subset of Y.

It is well known that Y equipped with this norm is a conditionally complete Banach lattice. Some examples of conditionally complete Banach lattices were given in [10].

The set $Y^+ := \{x \in Y : x \succeq 0\}$ is called the positive cone of Y, and its members are called the positive elements of Y. Clearly, the sum of two positive elements is again a positive element and that Y^+ is a closed convex cone in Y.

Throughout the paper we assume that Y is a continuous conditionally complete Banach lattice over the field of real numbers ($\mathbb{R}$) with the strong unit $\mathbf{1}$ equipped with the norm defined by (2), and with the minimal completion $\overline{Y}$ (see Definition 2.3). For definition of a continuous lattice see [7].

We denote by 0 the zero of both vector spaces X and Y under the operation " + ".

In the sequel, we give a definition of the vector valued decreasing inverse co-radiant functions. A version of the following definition for real valued functions was given in [8].

Definition 2.7. *A vector valued function $f : X \to \overline{Y}$ is called inverse co-radiant if $f(\gamma x) \preceq \frac{1}{\gamma} f(x)$ for all $x \in X$ and all $\gamma \in (0, 1]$.*

It is easy to see that f is inverse co-radiant if $f(\gamma x) \succeq \frac{1}{\gamma} f(x)$ for all $x \in X$ and all $\gamma \geq 1$.

Definition 2.8. *A vector valued function $f : X \to \overline{Y}$ is called decreasing if $x \leq y$ implies $f(x) \succeq f(y)$.*

Definition 2.9. *A vector valued function $f : X \to \overline{Y}$ is called decreasing inverse co-radiant if f is a decreasing and inverse co-radiant function.*

In this paper, we study vector valued decreasing inverse co-radiant functions $f : X \to \overline{Y}$ such that

$$0 \in \operatorname{dom} f := \{x \in X : f(x) \neq \top\}.$$

The following definition for real valued functions has been given in [14].

Definition 2.10. *Let $H := \{h : X \to Y : h \text{ is a vector valued function}\}$ and $f : X \to \overline{Y}$ be a vector valued function.*

(1) *The support set (or the set of all H-minorants) of f with respect to H is defined by*

$$\operatorname{supp}(f, H) := \{h \in H : h(x) \preceq f(x), \forall x \in X\}.$$

(2) *The function $f : X \to \overline{Y}$ is called abstract convex with respect to H (or*

H-convex) if there exists a subset U of H such that f is the upper envelope of this set:

$$f(x) = \sup_{h \in U} h(x), \quad (x \in X).$$

(3) *The subdifferential of the function $f : X \to \overline{Y}$ at a point $x_0 \in domf$ with respect to H (or H-subdifferential of f) is defined by*

$$\partial_H f(x_0) := \{h \in H : h(x) - h(x_0) \preceq f(x) - f(x_0), \ \forall \ x \in X\}.$$

The set H in Definition 2.10 is called the set of elementary functions.

3. Abstract Convexity of Vector Valued Decreasing Inverse Co-radiant Functions

In this section, we discuss on the abstract convexity of vector valued decreasing inverse co-radiant functions with respect to a certain class of elementary decreasing inverse co-radiant functions.

Now, consider the function $v : X \times X \times (Y^+ \setminus \{0\}) \to Y^+$ is defined by:

$$v(x, y, \alpha) := \sup\{\lambda \in Y^+ : 0 \preceq \lambda \preceq \alpha, \ \|\lambda\|x \leq y\}, \ \forall x, y \in X, \forall \alpha \in Y^+ \setminus \{0\}, \tag{4}$$

(with the convention $\sup \emptyset = 0$).

Remark 3.1. *Note that, for each $x, y \in X$ and $\alpha \in Y^+ \setminus \{0\}$, the set $D_{x,y,\alpha} := \{\lambda \in Y^+ : 0 \preceq \lambda \preceq \alpha, \ \|\lambda\|x \leq y\}$ is closed in Y^+. That is, if there exists a sequence $\{\lambda_n\}_{n \geq 1} \subset D_{x,y,\alpha}$ such that $\|\lambda_n - \lambda\| \longrightarrow 0$ as $n \to +\infty$ for some $\lambda \in Y^+$, then, $\lambda \in D_{x,y,\alpha}$.*
To this end, let $\{\lambda_n\}_{n \geq 1} \subset D_{x,y,\alpha}$ be such that $\|\lambda_n - \lambda\| \longrightarrow 0$ as $n \to +\infty$ for some $\lambda \in Y^+$. Since $\{\lambda_n\}_{n \geq 1} \subset D_{x,y,\alpha}$, it follows that $0 \preceq \lambda_n \preceq \alpha$ and $\|\lambda_n\|x \leq y$ for all $n \geq 1$, and so, $y - \|\lambda_n\|x \in S$ for all $n \geq 1$. This together with $\|\lambda_n\| \longrightarrow \|\lambda\|$ and the fact that S is closed implies that $y - \|\lambda\|x \in S$. Then, $\|\lambda\|x \leq y$.
On the other hand, we have $0 \preceq \lambda_n \preceq \alpha$ for all $n \geq 1$. Therefore, $(\alpha - \lambda_n)^- = 0$ for all $n \geq 1$, and hence,

$$\|(\alpha - \lambda_n)^-\| = 0, \ \forall \ n \geq 1. \tag{5}$$

Since

$$|(\alpha - \lambda_n)^- - (\alpha - \lambda)^-| \leq |\lambda_n - \lambda|, \ \forall \ n \geq 1,$$

it follows from (1) that

$$\|(\alpha - \lambda_n)^- - (\alpha - \lambda)^-\| \leq \|\lambda_n - \lambda\|, \ \forall \ n \geq 1. \tag{6}$$

Now, since $\|\lambda_n - \lambda\| \to 0$, we conclude from (5) and (6) that $\|(\alpha - \lambda)^-\| = 0$. This implies that $\lambda \preceq \alpha$. Hence, $\lambda \in D_{x,y,\alpha}$.

In the following, we give some properties of the function v.

Proposition 3.1. *For every* $x,\ y,\ x',\ y' \in X$; $\gamma \in (0,1]$; $\alpha,\ \alpha' \in Y^+ \setminus \{0\}$, $\beta > 0$, *one has*

$$x \leq x' \Rightarrow v(x,y,\alpha) \succeq v(x',y,\alpha), \tag{7}$$

$$y \leq y' \Rightarrow v(x,y,\alpha) \preceq v(x,y',\alpha), \tag{8}$$

$$\alpha \preceq \alpha' \Rightarrow v(x,y,\alpha) \preceq v(x,y,\alpha'), \tag{9}$$

$$v(\gamma x,y,\alpha) \preceq \frac{1}{\gamma} v(x,y,\alpha), \tag{10}$$

$$v(x,\gamma y,\alpha) \succeq \gamma v(x,y,\alpha), \tag{11}$$

$$v(\beta x,y,\alpha) = \frac{1}{\beta} v(x,y,\alpha\beta), \tag{12}$$

$$v(x,\beta y,\alpha) = \beta v(x,y,\frac{\alpha}{\beta}), \tag{13}$$

$$v(x,y,\alpha) = \alpha \iff \|\alpha\| x \leq y, \tag{14}$$

$$v(x,y,\alpha) \preceq \alpha, \tag{15}$$

$$\|v(x,y,\alpha)\| x \leq y,\ \textit{whenever}\ v(x,y,\alpha) \neq 0. \tag{16}$$

Proof. We only prove (10). Indeed, we have:

$$\begin{aligned} v(\gamma x,y,\alpha) &= \sup\{\lambda : 0 \preceq \lambda \preceq \alpha,\ \|\lambda\|\gamma x \leq y\} \\ &= \sup\{\lambda : 0 \preceq \lambda \preceq \alpha,\ \|\lambda\gamma\| x \leq y\} \\ &= \sup\{\frac{\beta}{\gamma} : 0 \preceq \frac{\beta}{\gamma} \preceq \alpha,\ \|\beta\| x \leq y\} \\ &= \frac{1}{\gamma}\sup\{\beta : 0 \preceq \beta \preceq \alpha\gamma,\ \|\beta\| x \leq y\} \\ &\preceq \frac{1}{\gamma}\sup\{\beta : 0 \preceq \beta \preceq \alpha,\ \|\beta\| x \leq y\} \\ &= \frac{1}{\gamma} v(x,y,\alpha). \end{aligned}$$

□

Remark 3.2. *In view of (7) and (10), it is worth noting that the function* $v(.,y,\alpha)$ *is decreasing inverse co-radiant for each* $y \in X$ *and each* $\alpha \in Y^+ \setminus \{0\}$.

Now, for each $y \in X$ and each $\alpha \in Y^+ \setminus \{0\}$, define the function $v_{(y,\alpha)} : X \to Y^+$ by $v_{(y,\alpha)}(x) = v(x,y,\alpha)$ for all $x \in X$.
Note that in view of Remark 3.2, one has each function $v_{(y,\alpha)}$ is decreasing inverse co-radiant. Let

$$L_0 := \{v_{(y,\alpha)} : y \in X,\ \alpha \in Y^+ \setminus \{0\}\}.$$

L_0 is called a set of elementary functions.

In the following, we give a characterization of vector valued decreasing inverse co-radiant functions with respect to L_0 (the set of elementary functions).

Theorem 3.1. *Let $f : X \to Y^+$ be a vector valued function. Then the following assertions are equivalent:*
(i) f is decreasing inverse co-radiant.
(ii) $\gamma f(y) \preceq f(x)$ for all $x, y \in X$ and all $\gamma \in (0, 1]$ such that $\gamma x \leq y$.
(iii) $\|v_{(y,\alpha)}(x)\| f(\frac{y}{\|\alpha\|}) \preceq \|\alpha\| f(x)$ for all $x, y \in X$ and all $\alpha \in Y^+ \setminus \{0\}$.

Proof. $(i) \Longrightarrow (ii)$. It is obvious.
$(ii) \Longrightarrow (iii)$. Let $\alpha \in Y^+ \setminus \{0\}$ and $x, y \in X$ be arbitrary. Clearly, (iii) holds if $v_{(y,\alpha)}(x) = 0$. Suppose that $v_{(y,\alpha)}(x) \neq 0$. Then, by (15) and (16), $0 \prec v_{(y,\alpha)}(x) \preceq \alpha$ and $\|v_{(y,\alpha)}(x)\| x \leq y$. It follows from (1) that $0 < \frac{\|v_{(y,\alpha)}(x)\|}{\|\alpha\|} \leq 1$. Thus, by the hypothesis (ii) and the fact that $\frac{\|v_{(y,\alpha)}(x)\|}{\|\alpha\|} x \leq \frac{y}{\|\alpha\|}$, we conclude that $\frac{\|v_{(y,\alpha)}(x)\|}{\|\alpha\|} f(\frac{y}{\|\alpha\|}) \preceq f(x)$. Therefore, (iii) holds.
$(iii) \Longrightarrow (i)$. Now, let $x \leq y$, then, by (14), $v_{(y,\frac{\alpha}{\|\alpha\|})}(x) = \frac{\alpha}{\|\alpha\|}$. Thus, in view of the hypothesis (iii), one has $f(y) = \|v_{(y,\frac{\alpha}{\|\alpha\|})}(x)\| f(y) \preceq f(x)$. So, f is decreasing. Now, let $\gamma \in]0, 1]$ and $x \in X$ be arbitrary. Therefore, it follows from (11) and (14) that

$$\gamma \frac{\alpha}{\|\alpha\|} = \gamma v_{(x,\frac{\alpha}{\|\alpha\|})}(x) \preceq v_{(\gamma x,\frac{\alpha}{\|\alpha\|})}(x).$$

This together with (1) implies that

$$\gamma \leq \|v_{(\gamma x,\frac{\alpha}{\|\alpha\|})}(x)\|. \tag{17}$$

So, by the hypothesis (iii) and (17) and the fact that Y^+ is a cone, we deduce that

$$\gamma f(\gamma x) \preceq \|v_{(\gamma x,\frac{\alpha}{\|\alpha\|})}(x)\| f(\gamma x) \preceq f(x).$$

Hence, f is inverse co-radiant, which implies that (i) holds. □

Now, we are going to show that each vector valued decreasing inverse co-radiant function is supremally generated by a certain class of decreasing inverse co-radiant functions.
In the sequel, for each $y \in X$ and each $\alpha \in Y^+ \setminus \{0\}$, define the function $u_{(y,\alpha)} : X \to Y^+$ by

$$u_{(y,\alpha)}(x) := \frac{\alpha}{\|\alpha\|} \|v_{(y,\alpha)}(x)\|, \ \forall\, x \in X. \tag{18}$$

Remark 3.3. *In view of (1) and Remark 3.2 and the fact that Y^+ is a cone, it is easy to check that each function $u_{(y,\alpha)}$ is decreasing inverse co-radiant and satisfies (7), (8) and (10)-(16).*

Let

$$L := \{u_{(y,\alpha)} : y \in X, \ \alpha \in Y^+ \setminus \{0\}\}. \tag{19}$$

We call L the set of elementary functions.

Lemma 3.1. *Let $\alpha, \alpha' \in Y^+ \setminus \{0\}$ be such that $\alpha \preceq \alpha'$, and let $y \in X$ be arbitrary. Then,*

$$u_{(y,\alpha)}(x) \preceq \frac{\|\alpha'\|}{\|\alpha\|} u_{(y,\alpha')}(x), \ \forall \ \in X.$$

Proof. Since $\alpha \preceq \alpha'$, it follows from (1) that $\|\alpha\| \leq \|\alpha'\|$, and by (9), one has $v_{(y,\alpha)}(x) \preceq v_{(y,\alpha')}(x)$ for all $x \in X$. So, again, in view of (1), we have

$$\|v_{(y,\alpha)}(x)\| \leq \|v_{(y,\alpha')}(x)\|, \ \forall \ x \in X. \tag{20}$$

This together with (18) and (20) and the fact that Y^+ is a cone implies that

$$\begin{aligned} u_{(y,\alpha')}(x) &= \frac{\alpha'}{\|\alpha'\|}\|v_{(y,\alpha')}(x)\| \\ &\succeq \frac{\alpha'}{\|\alpha'\|}\|v_{(y,\alpha)}(x)\| \\ &\succeq \frac{\alpha}{\|\alpha'\|}\|v_{(y,\alpha)}(x)\| \\ &= \frac{\|\alpha\|}{\|\alpha'\|}\frac{\alpha}{\|\alpha\|}\|v_{(y,\alpha)}(x)\| \\ &= \frac{\|\alpha\|}{\|\alpha'\|}u_{(y,\alpha)}(x), \ \forall \ x \in X, \end{aligned}$$

which completes the proof. □

Theorem 3.2. *Let $f : X \to Y^+ \cup \{\top\}$ be a vector valued function. Then, f is decreasing inverse co-radiant if and only if there exists a set $\triangle \subseteq L$ such that*

$$f(x) = \sup_{u_{(y,\alpha)} \in \triangle} u_{(y,\alpha)}(x), \ (x \in X). \tag{21}$$

In this case, one can take $\triangle := \{u_{(y,\alpha)} \in L : \alpha \preceq f(\frac{y}{\|\alpha\|})\}$. Hence, the vector valued function f is decreasing inverse co-radiant if and only if f is L-convex.

Proof. Suppose that (21) holds. Then, in view of Remark 3.3, f is decreasing inverse co-radiant. Conversely, assume that f is decreasing inverse co-radiant. Thus, according to Theorem 3.1 (the implication $(i) \Longrightarrow (iii)$), we have

$$\|v_{(y,\alpha)}(x)\| f(\frac{y}{\|\alpha\|}) \preceq \|\alpha\| f(x), \ \forall \ x, y \in X, \ \forall \ \alpha \in Y^+ \setminus \{0\}. \tag{22}$$

Now, fix $x \in X$, and let $u_{(y,\alpha)} \in \triangle$ be arbitrary. Then, it follows from (22) that $\alpha\|v_{(y,\alpha)}(x)\| \preceq \|\alpha\| f(x)$, and hence,

$$\frac{\alpha}{\|\alpha\|}\|v_{(y,\alpha)}(x)\| \preceq f(x).$$

Thus, in view of (18), we have

$$u_{(y,\alpha)}(x) \preceq f(x), \ \forall \ u_{(y,\alpha)} \in \triangle. \tag{23}$$

Now, Consider three possible cases.
Case (i). Suppose that $f(x) \neq 0, \top$. Let $\alpha_0 = f(x)$ and $y_0 = \|f(x)\|x$. Then, $\alpha_0 = f(\frac{y_0}{\|\alpha_0\|})$, and so, $u_{(y_0,\alpha_0)} \in \triangle$. Also, in view of (13) and (14), we have

$$u_{(y_0,\alpha_0)}(x) = \frac{\alpha_0}{\|\alpha_0\|}\|v_{(y_0,\alpha_0)}(x)\| = f(x). \tag{24}$$

Thus, it follows from (23) and (24) that

$$f(x) = \sup_{u_{(y,\alpha)} \in \triangle} u_{(y,\alpha)}(x).$$

Case (ii). If $f(x) = 0$. Assume that $u_{(y,\alpha)} \in \triangle$ is such that $u_{(y,\alpha)}(x) \neq 0$. In view of (18), one has

$$\|v_{(y,\alpha)}(x)\| = \|u_{(y,\alpha)}(x)\| > 0. \tag{25}$$

According to (22), we have $\|v_{(y,\alpha)}(x)\| f(\frac{y}{\|\alpha\|}) \preceq \|\alpha\| f(x) = 0$. This together with (25) and the fact that Y^+ is a cone implies that $f(\frac{y}{\|\alpha\|}) = 0$. Since $u_{(y,\alpha)} \in \triangle$, we conclude that $\alpha \preceq f(\frac{y}{\|\alpha\|}) = 0$, which contradicts $\alpha \in Y^+ \setminus \{0\}$. So, $u_{(y,\alpha)}(x) = 0$ for all $u_{(y,\alpha)} \in \triangle$. Therefore,

$$f(x) = 0 = \sup_{u_{(y,\alpha)} \in \triangle} u_{(y,\alpha)}(x).$$

Case (iii). Assume that $f(x) = \top$. Let $\gamma \in \mathbb{R}$ be such that $\gamma > 0$. Put $y_\gamma = \gamma x \in X$ and $\alpha_\gamma = \gamma \mathbf{1} \in Y^+ \setminus \{0\}$. Then, $f(\frac{y_\gamma}{\|\alpha_\gamma\|}) = f(x) = \top \succ \alpha_\gamma$. Hence, $u_{(y_\gamma,\alpha_\gamma)} \in \triangle$ for all $\gamma > 0$. Also, in view of (13) and (14), it is easy to check that $u_{(y_\gamma,\alpha_\gamma)}(x) = \gamma \mathbf{1}$ for all $\gamma > 0$. Therefore, by (23) , one has

$$\begin{aligned} f(x) &= \top \\ &= \sup_{\gamma > 0} \gamma \mathbf{1} \\ &= \sup_{u_{(y_\gamma,\alpha_\gamma)}} u_{(y_\gamma,\alpha_\gamma)}(x) \\ &\preceq \sup_{u_{(y,\alpha)} \in \triangle} u_{(y,\alpha)}(x) \\ &\preceq f(x), \end{aligned}$$

which completes the proof. □

4. Characterizations of Support Set and Subdifferential of Vector Valued Decreasing Inverse Co-radiant Functions

In this section, we present a description of the support set and the L-subdifferential of a vector valued decreasing inverse co-radiant function $f : X \to Y^+ \cup \{\top\}$. Let L be the set of elementary functions defined by (19).

Recall [14] that for a function $f : X \to Y^+ \cup \{\top\}$, the support set of f with respect to L is defined by:

$$\mathrm{supp}(f, L) := \{u_{(y,\alpha)} \in L : u_{(y,\alpha)}(x) \preceq f(x),\ \forall\, x \in X\}.$$

The following result gives us a characterization of the support set of a vector valued decreasing inverse co-radiant function $f : X \to Y^+ \cup \{\top\}$.

Proposition 4.1. *Let $f : X \to Y^+ \cup \{\top\}$ be a vector valued decreasing inverse co-radiant function. Then,*

$$\text{supp}(f, L) = \{u_{(y,\alpha)} \in L : \alpha \preceq f(\frac{y}{\|\alpha\|})\}.$$

Proof. Let $u_{(y,\alpha)} \in \text{supp}(f, L)$ be arbitrary. Then, $u_{(y,\alpha)}(x) \preceq f(x)$ for all $x \in X$, and so, for $x := \frac{y}{\|\alpha\|}$, it follows from (12), (14) and (18) that $\alpha = u_{(y,\alpha)}(\frac{y}{\|\alpha\|}) \preceq f(\frac{y}{\|\alpha\|})$. Conversely, suppose that $u_{(y,\alpha)} \in L$ is such that $\alpha \preceq f(\frac{y}{\|\alpha\|})$. According to Theorem 3.1 (the implication $(i) \implies (iii)$), for $v_{(y,\alpha)}$ (corresponding to $u_{(y,\alpha)}$), we have

$$\|v_{(y,\alpha)}(x)\| f(\frac{y}{\|\alpha\|}) \preceq \|\alpha\| f(x), \ \forall\, x \in X. \tag{26}$$

Thus, in view of (26) and the fact that $\alpha \preceq f(\frac{y}{\|\alpha\|})$, we conclude that $\alpha\|v_{(y,\alpha)}(x)\| \preceq \|\alpha\| f(x)$ for all $x \in X$, and hence,

$$\frac{\alpha}{\|\alpha\|}\|v_{(y,\alpha)}(x)\| \preceq f(x), \ \forall\, x \in X.$$

Then, by (18), one has

$$u_{(y,\alpha)}(x) \preceq f(x), \ \forall\, x \in X,$$

which completes the proof. □

Recall [14] that for a function $f : X \to Y^+ \cup \{\top\}$, the L-subdifferential of f at a point $x_0 \in X$ with $f(x_0) \in Y^+$, is defined as follows:

$$\partial_L f(x_0) := \{u_{(y,\alpha)} \in L : u_{(y,\alpha)}(x) - u_{(y,\alpha)}(x_0) \preceq f(x) - f(x_0), \ \forall\, x \in X\},$$

and for $f(x_0) = \top$, define $\partial_L f(x_0) := \emptyset$.

In the following, we give a description of the L-subdifferential of a vector valued decreasing inverse co-radiant function $f : X \to Y^+ \cup \{\top\}$ with respect to the elementary set L.

Proposition 4.2. *Let $f : X \to Y^+ \cup \{\top\}$ be a vector valued decreasing inverse co-radiant function, and let $x_0 \in X$ be such that $f(x_0) \neq 0, \top$. Then,*

$$\{u_{(y,\alpha)} \in L : \alpha \preceq f(\frac{y}{\|\alpha\|}),\, u_{(y,\alpha)}(x_0) = f(x_0)\} \subseteq \partial_L f(x_0).$$

Moreover, $\partial_L f(x_0) \neq \emptyset$.

Proof. Let

$$u_{(y,\alpha)} \in \{u_{(y,\alpha)} \in L : \alpha \preceq f(\frac{y}{\|\alpha\|}),\, u_{(y,\alpha)}(x_0) = f(x_0)\}$$

be arbitrary. In view of Proposition 4.1, one has $\alpha \preceq f(\frac{y}{\|\alpha\|})$ if and only if $u_{(y,\alpha)}(x) \preceq f(x)$ for all $x \in X$. This together with the fact that $u_{(y,\alpha)}(x_0) =$

$f(x_0)$ implies that $u_{(y,\alpha)} \in \partial_L f(x_0)$. Moreover, put $y := \|f(x_0)\|x_0$ and $\alpha := f(x_0)$. Therefore, it follows from (13), (14) and (18) that $u_{(y,\alpha)}(x_0) = f(x_0)$ and $f(\frac{y}{\|\alpha\|}) = \alpha$. Hence, $u_{(y,\alpha)} \in \partial_L f(x_0)$. ☐

Theorem 4.1. *Let $f : X \to Y^+ \cup \{\top\}$ be a vector valued decreasing inverse co-radiant function, and let $x_0 \in X$ be such that $f(x_0) \neq \top$. Then,*

$$\{u_{(y,\alpha)} \in L : f(x_0) \preceq u_{(y,\alpha)}(x_0),\ \alpha - u_{(y,\alpha)}(x_0) \preceq f(\frac{y}{\|\alpha\|}) - f(x_0)\} \subseteq \partial_L f(x_0).$$

Moreover, the equality holds if and only if $\inf_{x\in X} f(x) = 0$.

Proof. Let

$$\Delta := \{u_{(y,\alpha)} \in L : f(x_0) \preceq u_{(y,\alpha)}(x_0),\ \alpha - u_{(y,\alpha)}(x_0) \preceq f(\frac{y}{\|\alpha\|}) - f(x_0)\},$$

and $u_{(y,\alpha)} \in \Delta$ be arbitrary. First, note that in view of (15), we have

$$0 \preceq v_{(y,\alpha)}(x) \preceq \alpha$$

for all $x \in X$. Thus, by (1),

$$0 \leq \frac{\|v_{(y,\alpha)}(x)\|}{\|\alpha\|} \leq 1$$

for all $x \in X$. This together with $u_{(y,\alpha)} \in \Delta$ and the fact that Y^+ is a cone implies that

$$\begin{aligned}\frac{\|v_{(y,\alpha)}(x)\|}{\|\alpha\|}(\alpha - f(\frac{y}{\|\alpha\|})) &\preceq \frac{\|v_{(y,\alpha)}(x)\|}{\|\alpha\|}(u_{(y,\alpha)}(x_0) - f(x_0)) \\ &\preceq u_{(y,\alpha)}(x_0) - f(x_0),\ \forall\, x \in X. \qquad (27)\end{aligned}$$

According to Theorem 3.1 (the implication $(i) \Longrightarrow (iii)$), we have

$$\frac{\|v_{(y,\alpha)}(x)\|}{\|\alpha\|} f(\frac{y}{\|\alpha\|}) \preceq f(x),\ \forall\, x \in X. \qquad (28)$$

This together with (27) implies that

$$\begin{aligned}\frac{\alpha\|v_{(y,\alpha)}(x)\|}{\|\alpha\|} - f(x) &\preceq \frac{\|v_{(y,\alpha)}(x)\|}{\|\alpha\|}(\alpha - f(\frac{y}{\|\alpha\|})) \\ &\preceq u_{(y,\alpha)}(x_0) - f(x_0),\ \forall\, x \in X. \qquad (29)\end{aligned}$$

This implies that

$$\frac{\alpha\|v_{(y,\alpha)}(x)\|}{\|\alpha\|} - u_{(y,\alpha)}(x_0) \preceq f(x) - f(x_0),\ \forall\, x \in X. \qquad (30)$$

Now, by using (18) and (30), we conclude that

$$u_{(y,\alpha)}(x) - u_{(y,\alpha)}(x_0) \preceq f(x) - f(x_0),\ \forall\, x \in X,$$

and hence, $u_{(y,\alpha)} \in \partial_L f(x_0)$.

Now, suppose that $\inf_{x\in X} f(x) = 0$, and $u_{(y,\alpha)} \in \partial_L f(x_0)$ is arbitrary. So, by the definition,

$$u_{(y,\alpha)}(x) - u_{(y,\alpha)}(x_0) \preceq f(x) - f(x_0),\ \forall\, x \in X. \tag{31}$$

Since $0 \preceq u_{(y,\alpha)}(x)$ for all $x \in X$, it follows from (31) that

$$-u_{(y,\alpha)}(x_0) \preceq u_{(y,\alpha)}(x) - u_{(y,\alpha)}(x_0) \preceq f(x) - f(x_0),\ \forall\, x \in X.$$

Thus, $f(x_0) - u_{(y,\alpha)}(x_0) \preceq \inf_{x\in X} f(x) = 0$, which implies that $f(x_0) \preceq u_{(y,\alpha)}(x_0)$. Moreover, by putting $x := \frac{y}{\|\alpha\|}$ in (31), we obtain from (12), (14) and (18),

$$\alpha - u_{(y,\alpha)}(x_0) \preceq f(\frac{y}{\|\alpha\|}) - f(x_0).$$

Hence, $u_{(y,\alpha)} \in \Delta$.

In the sequel, we show that if $\Delta = \partial_L f(x_0)$, then, $\inf_{x\in X} f(x) = 0$. Let $\alpha \in Y^+ \setminus \{0\}$ be arbitrary such that $\alpha \succ f(0) - \inf_{x\in X} f(x)$. We claim that $u_{(0,\alpha)} \in \partial_L f(0)$. To this end, note that in view of (4), one has

$$v_{(0,\alpha)}(x) := \begin{cases} \alpha, & x \in -S \\ 0, & x \notin -S, \end{cases}$$

So, it follows from (18) that

$$u_{(0,\alpha)}(x) := \begin{cases} \alpha, & x \in -S \\ 0, & x \notin -S. \end{cases} \tag{32}$$

Now, let $x \in -S$. Since f is decreasing, so, $f(0) \preceq f(x)$. Then, by (32),

$$u_{(0,\alpha)}(x) - u_{(0,\alpha)}(0) = \alpha - \alpha = 0 \preceq f(x) - f(0),\ \forall\, x \in -S.$$

On the other hand, since $\alpha \succ f(0) - \inf_{x\in X} f(x)$, it follows from (32) that

$$u_{(0,\alpha)}(x) - u_{(0,\alpha)}(0) = 0 - \alpha \preceq f(x) - f(0),\ \forall\, x \in X \setminus (-S).$$

Therefore, $u_{(0,\alpha)} \in \partial_L f(0)$ for all $\alpha \in Y^+ \setminus \{0\}$ with $\alpha \succ f(0) - \inf_{x\in X} f(x)$. Moreover, since $\partial_L f(0) = \Delta$, we conclude that

$$f(0) \preceq u_{(0,\alpha)}(0) = \alpha,\ \forall\, \alpha \in Y^+ \setminus \{0\} \text{ with } \alpha \succ f(0) - \inf_{x\in X} f(x).$$

Now, let $\alpha \to [f(0) - \inf_{x\in X} f(x)]$, we conclude that $\inf_{x\in X} f(x) = 0$. □

Corollary 4.1. *Let $f : X \to Y^+ \cup \{\top\}$ be a vector valued decreasing inverse co-radiant function, and let $x_0 \in X$ be such that $f(x_0) \neq \top$. Define the function $g : X \to Y^+ \cup \{\top\}$ by $g(x) := f(x) - \inf_{x\in X} f(x)$ for all $x \in X$. Suppose that g is a decreasing inverse co-radiant function. Then,*

$$\begin{aligned} \partial_L f(x_0) &= \{u_{(y,\alpha)} \in L : f(x_0) \preceq u_{(y,\alpha)}(x_0) + \inf_{x\in X} f(x),\ \alpha - u_{(y,\alpha)}(x_0) \\ &\preceq f(\frac{y}{\|\alpha\|}) - f(x_0)\}. \end{aligned}$$

Proof. Since $\inf_{x\in X} g(x) = 0$ and $\partial_L g(x_0) = \partial_L f(x_0)$, so the result follows from Theorem 4.1. □

5. Conclusions

We first introduced and studied a new class of elementary functions, and by using this class, we characterized vector valued decreasing inverse co-radiant functions in a framework of abstract convexity. Moreover, we showed that the class of vector valued decreasing inverse co-radiant functions is abstract convex with respect to this class of elementary functions. Finally, as an application, we presented characterizations of the support set and the subdifferential of this class of functions. Decreasing inverse co-radiant functions have many applications in mathematical economics and optimization problems [6, 8, 9].

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