KB-OPERATORS ON BANACH LATTICES AND THEIR RELATIONSHIPS WITH DUNFORD-PETTIS AND ORDER WEAKLY COMPACT OPERATORS

Akbar Bahramnezhad[1], Kazem Haghnejad Azar[2]

Aqzzouz, Moussa and Hmichane proved that an operator T from a Banach lattice E into a Banach space X is b-weakly compact if and only if $\{Tx_n\}_n$ is norm convergent for every positive increasing sequence $\{x_n\}_n$ of the closed unit ball B_E of E. In the present paper, we introduce and study new classes of operators that we call KB-operators and WKB-operators. A continuous operator T from a Banach lattice E into a Banach space X is said to be KB-operator (respectively, WKB-operator) if $\{Tx_n\}_n$ has a norm (respectively, weak) convergent subsequence in X for every positive increasing sequence $\{x_n\}_n$ in the closed unit ball B_E of E. We investigate the relationships between KB-operators (respectively, WKB-operators) and some other operators on Banach lattices spacial their relationships with Dunford-Pettis and order weakly compact operators.

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1. Introduction

Recall that a Riesz space E is an order vector space in which $\sup(x, y)$ (it is customary to write sometimes $x \vee y$ instead of $\sup(x, y)$) exists for every $x, y \in E$. Let E be a Riesz space. For each $x, y \in E$ with $x \leq y$, the set $[x, y] = \{z \in E : x \leq z \leq y\}$ is called an order interval. A subset of E is said to be order bounded if it is included in some order interval. An operator $T : E \to F$ between Riesz spaces is said to be order bounded if it maps each order bounded subset of E into order bounded subset of F. The collection of all order bounded operators from a Riesz space E into a Riesz space F will be denoted by $L_b(E, F)$. The collection of all order bounded linear functionals on a Riesz space E will be denoted by $E^\sim$, that is $E^\sim = L_b(E, \mathbb{R})$. A subset of a Riesz space E is b-order bounded if it is order bounded in $E^{\sim\sim} := (E^\sim)^\sim$. A Banach lattice E is a Banach space $(E, \|.\|)$ such that E is a Riesz space and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. A sequence $\{x_n\}_n$ in a Riesz space is said to be disjoint whenever $|x_n| \wedge |x_m| = 0$ holds for $n \neq m$. A Banach lattice E has order continuous norm if $\|x_\alpha\| \to 0$ for every decreasing net $(x_\alpha)_\alpha$ with $\inf_\alpha x_\alpha = 0$. If E is a Banach lattice, its topological dual E', endowed with the dual norm and dual order is also a Banach lattice. A Banach lattice E is is said to be an AM-space if for each $x, y \in E$ such that $|x| \wedge |y| = 0$, we have $\|x + y\| = max\{\|x\|, \|y\|\}$. A Banach lattice E is an AL-space if its topological dual E' is an AM-space. A Banach lattice E is said to be KB-space whenever each increasing norm bounded sequence of E^+ is norm convergent. An operator $T : E \to F$ between two Riesz spaces is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. Note that each positive linear

[1]Department of Mathematics, University of Mohaghegh Ardabili, Ardabil of Iran
bahramnezhad@uma.ac.ir
[2]Department of Mathematics, University of Mohaghegh Ardabili, Ardabil of Iran
haghnejad@uma.ac.ir

mapping on a Banach lattice is continuous. An operator T from a Banach space X into a Banach space Y is compact (resp. weakly compact) if $\overline{T(B_X)}$ is compact (resp. weakly compact) where B_X is the closed unit ball of X. A sequence $\{x_n\}_n$ in a normed space E is weakly convergent to $x \in E$ if for each $x' \in E'$, $x'(x_n) \to x'(x)$ in $\mathbb{R}$. For terminology concerning Banach lattice theory and positive operators, we refer the reader to the excellent book of [1].

Alpay-Altin-Tonyali introduced the class of b-weakly compact operators for Riesz spaces having separating order duals [2]. An operator $T : E \to X$, mapping each b-order bounded subset of E into a relatively weakly compact subset of X is called a b-weakly compact operator. Any Banach lattice is a Riesz space having separating order dual. They proved that a continuous operator T from a Banach lattice E into a Banach space X is b-weakly compact if and only if $\{Tx_n\}_n$ is norm convergent for each b-order bounded increasing sequence $\{x_n\}_n$ in E^+ if and only if $\{Tx_n\}_n$ is norm convergent to zero for each b-order bounded disjoint sequence $\{x_n\}_n$ in E^+ [3]. In [6], authors proved that an operator T from a Banach lattice E into a Banach space X is b-weakly compact if and only if $\{Tx_n\}_n$ is norm convergent for every positive increasing sequence $\{x_n\}_n$ of the closed unit ball B_E of E. The aim of this paper is to define new classes of operators on Banach lattices that we call KB-operators and WKB-operators, and study some of their properties. Our definitions is based on the notion of positive increasing norm bounded sequence.

Definition 1.1. A continuous operator T from a Banach lattice E into a Banach space X is said to be KB-operator if $\{Tx_n\}_n$ has a norm convergent subsequence in X for every positive increasing sequence $\{x_n\}_n$ in the closed unit ball B_E of E.

Definition 1.2. A continuous operator T from a Banach lattice E into a Banach space X is said to be WKB-operator if $\{Tx_n\}_n$ has a weak convergent subsequence in X for every positive increasing sequence $\{x_n\}_n$ in the closed unit ball B_E of E.

In [7], authors proved that if E and F are Banach lattices, then each b-weakly compact operator $T : E \to F$ admits a b-weakly compact adjoint T' if and only if E' or F' is a KB-space. They established that if E and F are Banach lattices such that the norm of E is order continuous, then each operator $T : E \to F$ is b-weakly compact whenever its adjoint T' is b-weakly compact if and only if E or F is a KB-space. As b-weakly compact operators [7], the class of KB-operators and WKB-operators does not satisfy duality property. In fact the identity operator of the Banach lattice ℓ^1 is a KB-operator (respectively, WKB-operator); but its adjoint which is the identity operator of the Banach lattice ℓ^∞, is not a KB-operator (respectively, WKB-operator). Conversely, the identity operator of the Banach lattice c_0 is not a KB-operator (respectively, not WKB-operator); but its adjoint, which is the identity operator of the Banach lattice ℓ^1, is a KB-operator (respectively, WKB-operator).

2. Main results

The collection of KB-operators and WKB-operators will be denoted by $L_{KB}(E, X)$ and $W_{KB}(E, X)$. The collection of b-weakly compact operators will be denoted by $W_b(E, X)$ and the collection of weakly compact and compact operators will be denoted by $W(E, X)$ and $K(E, X)$. Clearly $K(E, X) \subset W(E, X) \subset W_b(E, X) \subset L_{KB}(E, X) \subset W_{KB}(E, X)$. We will prove that if E is a KB-space, then $W_b(E, X) = L_{KB}(E, X)$ for each Banach space X.

Proposition 2.1. *Let E and F be Banach lattices and $T : E \to F$ be a positive operator. Then the following statements are equivalent.*

(1) T is b-weakly compact.
(2) T is KB-operator.
(3) T is WKB-operator.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.
For (3) $\Rightarrow$ (1) Let $T : E \to F$ be a positive WKB-operator and let $\{x_n\}_n$ be a positive increasing sequence in the closed unit ball B_E of E. By our hypothesis there exists a subsequence $\{Tx_{n_j}\}_j$ which $Tx_{n_j} \xrightarrow{w} x$, where x is an element of X. Now, by [1, Theorem 3.52], we have $Tx_{n_j} \xrightarrow{\|\cdot\|} x$. Since $\{Tx_n\}_n$ is an increasing sequence, $Tx_n \xrightarrow{\|\cdot\|} x$. Then T is b-weakly compact and we are done. □

Example 2.2. In the statement of the following example, c denotes the usual Banach lattice of convergent real sequences and c_0 denotes the subspace of null sequences. If for each $x = (x_1, x_2, x_3, ...) \in c$ we put $x_\infty = lim x_n$ then the operator $T : c \to c_0$ defined by $T(x) = (x_\infty, x_1 - x_\infty, x_2 - x_\infty, ...)$ is not a positive operator. Clearly the sequence $\{x_m\}_m$ defined by

$$x_m(n) = \begin{cases} \frac{1}{2} & m \leq n \\ 1 & m > n \end{cases}$$

is a positive increasing sequence in the closed unit ball of c. We claim that $\{Tx_m\}_m$ has no weak convergent subsequence. Indeed, note first that $Tx_m = (\frac{1}{2}, ..., \frac{1}{2}, 0, 0, ...)$, where the $\frac{1}{2}$'s occupy the first n positions, is an increasing sequence which is a weak Cauchy sequence but is not a norm Cauchy sequence in c_0 (see [1, P.233]). If $\{Tx_m\}_m$ has a weak convergent subsequence, then by [9, Proposition 1.4.1], $\{Tx_m\}_m$ is norm convergent which is a contradiction. Hence T is not KB-operator.

Note that each weakly compact operator is a KB-operator but the converse may be false in general. For example, the identity operator $I : L^1[0,1] \to L^1[0,1]$ is a KB-operator but is not weakly compact.

Proposition 2.3. *Let E and F be two Banach lattices such that the norm of E' is order continuous. Then each positive KB-operator $T : E \to F$ is weakly compact.*

Proof. Let $T : E \to F$ be a positive KB-operator. By using Proposition 2.1, T is b-weakly compact. Hence from [8, Theorem 2.3], T is weakly compact. □

Recall that a Banach space is said to have Schur property whenever every weak convergent sequence is norm convergent, i.e., whenever $x_n \xrightarrow{w} 0$ implies $\|x_n\| \to 0$. Let E, F be Banach lattices. If either E or F has the Schur property then $L(E, F) = W_b(E, F)$ [3].

Proposition 2.4. *Let E be a Banach lattice and X a Banach space with Schur property. Then every WKB-operator $T : E \to X$ is a KB-operator.*

Proof. Let $\{x_n\}_n$ be a positive increasing sequence in B_E. Since T is WKB-operator, there exists subsequence $\{Tx_{n_j}\}_j$ which is weakly convergent. Hence, by property Schur of X, $\{Tx_{n_j}\}_j$ is norm convergent. Then T is a KB-operator. □

Proposition 2.5. *The collection of all KB-operators from a Banach lattice E into a Banach space X is a norm closed subspace for the collection of all operators from E into X.*

Proof. We only show that $\overline{L_{KB}(E, X)} = L_{KB}(E, X)$. Let $S \in \overline{L_{KB}(E, X)}$. We have to show that S is a KB-operator. For each $\varepsilon > 0$, there exists $T \in L_{KB}(E, X)$ such that $\|S - T\| < \varepsilon$. Let $\{x_n\}_n$ be a positive increasing sequence in B_E. Since T is KB-operator, there exists subsequence $\{Tx_{n_j}\}_j$ of $\{Tx_n\}_n$ such that $Tx_{n_j} \xrightarrow{\|\cdot\|} x$ for an element $x \in X$. Since

$$\|Sx_{n_j} - x\| \leq \|Sx_{n_j} - Tx_{n_j}\| + \|Tx_{n_j} - x\| \leq \varepsilon(\|x\| + 1),$$

$Sx_{n_j} \xrightarrow{\|\cdot\|} x$. Then S is a KB-operator. □

Proposition 2.6. *Let E, F be Banach lattices and X a Banach space. Then we have the following assertions:*

(1) If $T \in L(F,X)$ and $S \in L_{KB}(E,F)$, then $TS \in L_{KB}(E,X)$. As a consequence, $L_{KB}(E)$ is a left ideal of $L(E)$.
(2) If $T \in L_{KB}(F,X)$ and $S \in L(E,F)^+$, then $TS \in L_{KB}(E,F)$. As a consequence, $L_{KB}(E)$ is a right ideal of $L(E)^+$.

Proof. (1) Let S be a KB-operator and $\{x_n\}_n$ be a positive increasing sequence in B_E. Then there exists subsequence $\{Sx_{n_j}\}_j$ which is norm convergent to an element $x \in F$. Since T is continuous, $\{TSx_{n_j}\}_j$ is norm convergent to Tx. Then TS is a KB-operator.
(2) Let T be a KB-operator and $\{x_n\}_n$ be a positive increasing sequence in B_E. Since S is positive, we may assume that $\{Sx_n\}_n$ is a positive increasing sequence in B_F. Then $\{TSx_n\}_n$ is a norm bounded and positive increasing sequence in F. Since T is KB-operator, $\{TSx_n\}_n$ has a norm convergent subsequence. Then TS is a KB-operator. This completes the proof.

□

Corollary 2.7. *Let E, F be Banach lattices, X a Banach space and $S : E \to F$ a positive operator and $T : F \to X$ be a continuous operator. If either S or T is a KB-operator, then TS is likewise a KB-operator.*

Proof. Let E, F be Banach lattices, X a Banach space and $S : E \to F$ a positive operator and $T : F \to X$ be a continuous operator. If S is a KB-operator then by part (1) of Proposition 2.6, TS is a KB-operator and if T is a KB-operator, then by part (2) of Proposition 2.6, TS is a KB-operator. □

We obtain the following result:

Corollary 2.8. *The space $L_{KB}(E)$ forms a two sided norm closed ideal in $L(E)^+$.*

Proof. By Proposition 2.5, The space $L_{KB}(E)$ is norm closed. Let $T \in L_{KB}(E)$ and $S \in L(E)^+$. By part (2) of Proposition 2.6, TS is a KB-operator, so, $L_{KB}(E)$ is a right ideal of $L(E)^+$ and by part (1) of Proposition 2.6, ST is a KB-operator. Therefore, $L_{KB}(E)$ is a left ideal of $L(E)^+$. This completes the proof. □

Recall from [2, Corollary 2.9] that if $S,T : E \to F$ are operators between Banach lattices with $0 \leq S \leq T$ and T is a b-weakly compact operator, then S is also a b-weakly compact operator. Now, we show that KB-operators satisfy domination property.

Proposition 2.9. *Let E and F be Banach lattices and $S,T : E \to F$ are operators with $0 \leq S \leq T$. If T is a KB-operator, then S is also a KB-operator.*

Proof. Let E and F be Banach lattices and $S,T : E \to F$ are operators with $0 \leq S \leq T$ and let T be a KB-operator. Since T is a positive KB-operator, by Proposition 2.1, T is b-weakly compact. So, by above argument, S is b-weakly compact and so, is a KB-operator. □

Remark 2.10. Similarly the positive WKB-operators satisfy domination property.

Recall that a Banach lattice E is said to be KB-space whenever each increasing norm bounded sequence of E^+ is norm convergent. Now we obtain the following result which is similar to [2, Proposition 2.10]:

Proposition 2.11. *Let E be a Banach lattice. E is a KB-space if and only if $I : E \to E$ is a KB-operator.*

Proof. Let E be a KB-space and $\{x_n\}_n$ be a positive increasing sequence in B_E. Then by our hypothesis, $\{x_n\}_n$ is norm convergent. So, $\{x_n\}_n = \{Ix_n\}_n$ is norm convergent. Then I is KB-operator.
Conversely, let $I : E \to E$ be a KB-operator and $\{x_n\}_n$ be an increasing norm bounded sequence in E^+. We may assume that $\{x_n\}_n$ is a positive increasing sequence in B_E. As I is KB-operator, $\{Ix_n\}_n$ has a norm convergent subsequence. On the other hand, since $\{x_n\}_n = \{Ix_n\}_n$ is an increasing sequence, $\{x_n\}_n$ is norm convergent. So, E is a KB-space. □

As a consequence of preceding proposition, we have the following result:

Corollary 2.12. *Let E be a Banach lattice. E is a KB-space if and only if $I : E \to E$ is a WKB-operator.*

Proposition 2.13. *Let E be a Banach lattice. Then the following statements are equivalent:*

(1) E is a KB-space.
(2) $L(E, X) = L_{KB}(E, X)$ for each Banach space X.

Proof. Let E be a KB-space, X be a Banach space and let T be a continuous operator from E into X, and $\{x_n\}_n$ be a positive increasing sequence in B_E. Since E is a KB-space, $\{x_n\}_n$ is norm convergent. Since T is a continuous operator, $\{Tx_n\}_n$ is norm convergent. So, T is a KB-operator. Then $L(E, X) \subset L_{KB}(E, X)$. On the other hand, $L_{KB}(E, X) \subset L(E, X)$. Hence $L_{KB}(E, X) = L(E, X)$. Conversely, we assume that $L_{KB}(E, X) = L(E, X)$ for every Banach space X. Then the identity operator $I : E \to E$ is a KB-operator. So by Proposition 2.11, E is a KB-space.

□

For the next two results we need the following lemmas which are just [5, Proposition 2.1] and [7, Corollary 2.3]:

Lemma 2.14. *Let E be a Banach lattice. Then the following statements are equivalent:*

(1) E is a KB-space.
(2) $L(E, X) = W_b(E, X)$ for each Banach space X.

Lemma 2.15. *Let F be a Banach lattice. Then the following statements are equivalent:*

(1) For any Banach lattice E, each operator from E into F is b-weakly compact.
(2) Each operator from c_0 into F is b-weakly compact (resp. compact).
(3) Each positive operator from c_0 into F is b-weakly compact (resp. compact).
(4) F is a KB-space.

Corollary 2.16. *Let E be a KB-space. Then $W_b(E, X) = L_{KB}(E, X)$ for each Banach space X.*

Proof. Let E be a KB-space. Then by Proposition 2.13 and Lemma 2.14, $L(E, X) = L_{KB}(E, X)$ and $W_b(E, X) = L(E, X)$. So, $W_b(E, X) = L_{KB}(E, X)$ for each Banach space X. This ends the proof. □

Corollary 2.17. *Let $T : E \to X$ be an operator from a Banach lattice E into a Banach space X. If T factors through a KB-space, then T is a KB-operator.*

Proof. Assume that T factors through a KB-space, i.e., there exist a KB-space F and two operators $Q : E \to F$, $S : F \to X$ such that $T = S \circ Q$. Let $\{x_n\}_n$ be a positive increasing sequence in B_E. Since F is a KB-space, by Lemma 2.15, Q is a KB-operator. Hence $\{Qx_n\}_n$ has a norm convergent subsequence. Then $\{S \circ Q(x_n)\}_n$ has a norm convergent subsequence. So, $T = S \circ Q$ is also a KB-operator. □

Let E be a Banach lattice, X a Banach space and $T : E \to X$ be a continuous operator. Then T is b-weakly compact if and only if $\{Tx_n\}_n$ is norm convergent to zero for every b-order bounded disjoint sequence $\{x_n\}_n \subset E^+$ if and only if $\{Tx_n\}_n$ is norm convergent in X for every positive increasing sequence $\{x_n\}_n$ in the closed unit ball B_E of E [3, 6].

Proposition 2.18. *(*[4, Proposition 1]*) Let E be a Banach lattice, X a Banach space and $T : E \to X$ be a continuous operator. Then the following assertions are equivalent:*
(1) *T is b-weakly compact.*
(2) *$\{Tx_n\}_n$ is norm convergent for every b-order bounded increasing sequence $\{x_n\}_n \subset E^+$.*

Corollary 2.19. *Let E, F be Banach lattices and $T : E \to F$ be a positive operator. Then the following assertions are equivalent:*
(1) *T is a KB-operator.*
(2) *$\{Tx_n\}_n$ is norm convergent to zero for every b-order bounded disjoint sequence $\{x_n\}_n \subset E^+$.*
(3) *$\{Tx_n\}_n$ is norm convergent for every b-order bounded increasing sequence $\{x_n\}_n \subset E^+$.*

An operator $T : E \to F$ between two Banach spaces is called a Dunford-Pettis operator whenever $x_n \xrightarrow{w} 0$ implies $Tx_n \xrightarrow{\|.\|} 0$. We show that each Dunford-Pettis operator is KB-operator. The converse is not always true. In fact, the identity operator of the Banach lattice ℓ^2 is KB-operator, but it is not Dunford-Pettis.
Recall that if E is a Banach lattice and if $0 \leqslant x'' \in E''$, then the principal ideal $I_{x''}$ generated by $x'' \in E''$ under the norm $\|.\|_\infty$ defined by

$$\|y''\|_\infty = \inf\{\lambda > 0 : \ |y''| \leq \lambda x''\}, \ y'' \in I_{x''},$$

is an AM-space with unit x'', whose closed unit ball is order interval $[-x'', x'']$ [1, Theorem 4.21].

Lemma 2.20. *Let E be a Banach lattice. Then every b-order bounded disjoint sequence in E is weakly convergent to zero.*

Proof. Let $\{x_n\}_n$ be a disjoint sequence in E such that $\{x_n\}_n \subseteq [-x'', x'']$ for some $x'' \in E''$. Let $Y = I_{x''} \cap E$ and equip Y with the order unit norm $\|.\|_\infty$ generated by x''. The space $(Y, \|.\|_\infty)$ is an AM-space. So, Y' is an AL-space and then its norm is order continuous. Now, by Theorem 2.4.14 from [9], we see that $x_n \xrightarrow{w} 0$. □

Proposition 2.21. *Every Dunford-Pettis operator from a Banach lattice E into a Banach space X is a KB-operator.*

Proof. Let T be a Dunford-Pettis operator from a Banach lattice E into a Banach space X. It is enough to show that $\{Tx_n\}_n$ is norm convergent to zero for each b-order bounded disjoint sequence $\{x_n\}_n$ in E^+. Let $\{x_n\}_n$ be a b-order bounded disjoint sequence in E^+. As the canonical embedding of E into E'' is a lattice homomorphism, $\{x_n\}_n$ is an order bounded disjoint sequence in E''. By using preceding lemma, $\{x_n\}_n$ is $\sigma(E, E')$ convergent to zero in E. Since T is Dunford-Pettis, $\{Tx_n\}_n$ is norm convergent to zero. This completes the proof. □

To give conditions under which a KB-operator is Dunford-Pettis, we will need the following lemma [6, Lemma 2.8].

Lemma 2.22. *Let E be a Banach lattice. Then every positive norm bounded net $\{x_\alpha\}_\alpha$ of E is b-order bounded, i.e., $\{x_\alpha\}_\alpha$ is order bounded in the topological bidual E''.*

Theorem 2.23. *Let F be a Banach lattice. Then each positive KB-operator from an AM-space E into F is Dunford-Pettis.*

Proof. Let F be a Banach lattice, E an AM-space and $T : E \to F$ be a positive KB-operator. Suppose that T is not Dunford-Pettis. Note that, for every $x \in E$, $\rho(x) = \|x\|$ is a continuous lattice seminorm on E. Since T is not Dunford-Pettis, there exists a sequence $\{x_n\}_n$ in E with $x_n \xrightarrow{w} 0$ and $\|Tx_n\| \geq 1$. By Theorem 4.31 from [1], E has weakly sequentially continuous lattice operations. So, we may assume that $\{x_n\} \subset E^+$. Now by Corollary 2.3.5 of [9], for every $0 < c < 1$, there exists a subsequence $\{k_n\}_n \subset \mathbb{N}$ and a disjoint sequence $\{y_n\}_n \subset E^+$ such that

$$y_n \leqslant x_{k_n},\ \|Ty_n\| \geqslant c$$

for all $n \in \mathbb{N}$. Since $y_n \leqslant x_{k_n}$ and $x_n \xrightarrow{w} 0$, the sequence $\{y_n\}$ is norm bounded. So, the sequence $u_n = \Sigma_{i=1}^n y_i$ is an increasing norm bounded sequence. Hence, from Lemma 2.22, there exists $x'' \in E''_+$ such that $0 \leqslant u_n \leqslant x''$. So, $\{u_n\}_n$ is a b-order bounded increasing sequence in E^+. Then by Corollary 2.19, $\{Tu_n\}_n$ is norm convergent. Since $y_n = u_n - u_{n-1}$, we have $\|Ty_n\| \to 0$, which is a contradiction. Hence T is Dunford-Pettis and we are done. □

Recall that an operator T from a Banach lattice E into a Banach space X is called o-weakly compact if for each order bounded subset A of E, $T(A)$ is a relatively weakly compact subset of X. The identity operator of the Banach lattice c_0 is an o-weakly compact operator, but is not a KB-operator (respectively, not a WKB-operator).

Proposition 2.24. *Let E be a Banach lattice, X a Banach space and $T : E \to X$ be a continuous operator. If $T'' : E'' \to X''$ is o-weakly compact, then T is WKB-operator.*

Proof. Let $\{x_n\}$ be a positive increasing sequence of the closed unit ball B_E of E. By Lemma 2.22, the set $A = \{x_n : n \in N\}$ is an order bounded subset of E''. So, by our hypothesis, $T''(A) = T(A)$ is a relatively weakly compact subset of X. Hence $\{Tx_n\}_n$ has a weakly convergent subsequence. Then T is WKB-operator. □

Recall that a continuous operator $T : X \to E$ from a Banach space into a Banach lattice is semicompact if for each $\varepsilon > 0$ there exists some $u \in E^+$ such that

$$T(U) \subseteq [-u, u] + \varepsilon V$$

where U and V denote the closed unit balls of X and E, respectively. Note that the identity operator of the Banach lattice ℓ^∞ is semicompact but is not KB-operator and the identity operator of the Banach lattice ℓ^2 is a KB-operator which is not semicompact.
As a consequence of [3], we obtain:

Corollary 2.25. *Let E, F be Banach lattices and $T : E \to F$ be a continuous operator. If $T' : F' \to E'$ is semicompact, then T is KB-operator.*

Recall that an ordered vector space E is a Riesz space if and only if the absolute value $|x| = x \vee (-x)$ exists for each vector $x \in E$ (see [1, P.7]). If E and F are Riesz spaces with F Dedekind complete, then the ordered vector space $L_b(E, F)$ is a Dedekind complete Riesz space [1, Theorem 1.18].

Remark 2.26. We now show that $L_{KB}(E, F)$ is not a Riesz space. For an operator $T : E \to F$ between two Riesz spaces we shall say that its modulus $|T|$ exists (or that T possesses a modulus) whenever $|T| := T \vee (-T)$ exists-in the sense that $|T|$ is the supremum of the set $\{-T, T\}$ in $L(E, F)$. This example due to Z.L. Chen and A.W. Wickstead in [10] shows that the order bounded KB-operators from a Banach lattice into a Dedekind complete Banach lattice do not form a lattice, i.e., a modulus of a KB-operator need not be a KB-operator. Let $E = C[0, 1]$, $F = l_\infty(F_n)$ where $F_n = (l_\infty, \|.\|)$ and $\|(\lambda_k)\| = max\{\|(\lambda_k)\|_\infty, nlimsup(|\lambda_k|)\}$ for all $(\lambda_k) \in l_\infty$. Then for each $n \in \mathbb{N}$, F_n is a Dedekind complete AM-space, hence so is F. Define $T_n : E \to F_n$ by $T_n(f) = (2^n. \int_{I_n} f.r_k dt)_{k=1}^\infty \in F_n$ for all $f \in E$, where r_n is the n'th Rademacher function on $[0, 1]$ and $I_n = (2^{-n}, 2^{-n+1})$.

Now define $T : E \to F$ by $T(f) = (\frac{1}{n}T_n(f))_{n=1}^{\infty}$. Then T is a weakly compact operator, so T is a KB-operator and its modulus $|T|$ exists and $|T|$ is not order weakly compact hence not b-weakly compact and by Proposition 2.1, not KB-operator. So, $L_{KB}(E, F)$ is not a lattice.

Problem 2.27. *Give an operator T from a Banach lattice E into a Banach space X which is a KB-operator; but is not b-weakly compact.*

Problem 2.28. *Give an operator T from a Banach lattice E into a Banach space X which is a WKB-operator; but is not KB-operator.*

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