

STRONGLY EXPONENTIALLY CONVEX FUNCTIONS

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In this paper, we define and introduce some new concepts of the strongly exponentially convex functions with respect to an auxiliary non-negative bifunction. We establish various new relationships among various concepts of strongly exponentially convex functions. We have also investigated the optimality conditions for the strongly exponentially convex functions. It is shown that the difference of strongly exponentially convex functions and strongly exponentially affine functions is again an exponentially convex function. Results obtained in this paper can be viewed as refinement and improvement of previously known results.

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1. Introduction

In recent years, several extensions and generalizations have been considered for classical convexity. Strongly convex functions were introduced and studied by Polyak [27], which play an important part in the optimization theory and related areas, see, for example, [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 15, 16, 17, 18, 19, 21, 26] and the references therein. Adamek [1] introduced an other class of convex function with respect to an arbitrary non-negative bifunction, called relative strongly convex functions. With appropriate choice of non-negative bifunction, one can obtain various known classes of convex functions. For the properties of the strongly convex functions, see Adamek [1], Nikodem et al. [2, 3, 4, 5, 6, 7], Awan et al. [5, 6, 7, 8] and Noor et al. [21].

It is known that more accurate and inequalities can be obtained using the algorithmically convex functions than the convex functions. Closely related to the log-convex functions, we have the concept of exponentially convex(concave) functions, which have important applications in information theory, big data analysis, machine learning and statistic, see, for example, [2, 24] and the references therein. Exponentially convex(concave) functions can be considered as a significant extension of the convex functions. Pal and Wong [24] have discussed its applications in information geometry and statistics. Antczak [3] introduced these exponentially convex functions implicitly and discussed their role in mathematical programming. Alirazaie and Mathur [2], Dragomir and Gomm [10, 12] and Noor and Noor [20, 22] have derived several results for these exponentially convex functions.

Inspired by the work of Adamek [1], Nikodem et al. [18] and Noor et al. [21], we introduce and consider another class of nonconvex functions with respect to an arbitrary non-negative bifunction. This class of nonconvex functions is called the strongly exponentially convex functions. Several new concepts of monotonicity are introduced. We establish the relationship between these classes and derive some new results under some mild conditions. We

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have also investigated the optimality conditions for the strongly exponentially convex functions. It is shown that the minimum of the differentiable exponentially convex function can be characterized by a class of variational inequalities, which is called exponential variational inequality. We prove that the difference of strongly exponentially convex functions and strongly exponentially affine functions is again an exponentially convex function. It is expected that the ideas and techniques of this paper may stimulate further research in this field.

2. Preliminary Results

Let K be a nonempty closed set in a real Hilbert space H . We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm, respectively. Let $F : K \rightarrow R$ be a continuous function. Let $G(\cdot, \cdot) : [0, \infty) \times [0, \infty) \rightarrow R$ be a non-negative bifunction.

Definition 2.1. [18]. *The set K in H is said to be a convex set, if*

$$u + t(v - u) \in K, \quad \forall u, v \in K, t \in [0, 1].$$

We now consider a class of exponentially convex function, which is mainly due to Antczak [3].

Definition 2.2. *A positive function F is said to be exponentially convex function, if*

$$e^{F((1-t)a+tb)} \leq (1-t)e^{F(a)} + te^{F(b)}, \quad \forall a, b \in K, \quad t \in [0, 1]. \quad (1)$$

or equivalently

Definition 2.3. *A positive function F is said to be exponentially convex function, if*

$$e^{F((1-t)a+tb)} \leq \log[(1-t)e^{F(a)} + te^{F(b)}], \quad \forall a, b \in K, \quad t \in [0, 1]. \quad (2)$$

One can also define the exponentially convex functions on $I = [a, b]$.

Definition 2.4. *Let $I = [a, b]$. Then F is exponentially convex function, if and only if,*

$$\begin{vmatrix} 1 & 1 & 1 \\ a & x & b \\ e^{F(a)} & e^{F(x)} & e^{F(b)} \end{vmatrix} \geq 0; \quad a \leq x \leq b.$$

The following statements are equivalent:

- (1) F is exponentially convex function.
- (2) $e^{F(x)} \leq e^{F(a)} + \frac{e^{F(b)} - e^{F(a)}}{b-a}(x-a)$.
- (3) $e^{F(x)} \leq \frac{b-x}{b-a}e^{F(a)} + \frac{x-a}{b-a}e^{F(b)}$.
- (4) $\frac{e^{F(x)} - e^{F(a)}}{x-a} \leq \frac{e^{F(b)} - e^{F(a)}}{b-a} \leq \frac{e^{F(b)} - e^{F(x)}}{b-x}$.
- (5) $(x-a)e^{F(a)} + (b-a)e^{F(x)} + (a-x)e^{F(b)} \geq 0$.
- (6) $\frac{e^{F(a)}}{(b-a)(a-x)} + \frac{e^{F(x)}}{(x-b)(a-x)} + \frac{e^{F(b)}}{(b-a)(x-b)} \geq 0$,

where $x = (1-t)a + tb \in [0, 1]$.

Remark 2.1. *If the exponentially convex function F is differentiable, then, from*

$$e^{F(x)} \leq e^{F(a)} + \frac{e^{F(b)} - e^{F(a)}}{b-a}(x-a),$$

we have

$$\langle F'(x)e^{F(x)}, b-a \rangle \leq e^{F(b)} - e^{F(a)},$$

where $F'(\cdot)$ is the differential of the function F .

For the applications of the exponentially convex(concave) functions in the mathematical programming and information theory, see Antczak [3] and Alirezaei and Mathar[2]. To convey an idea of the applications of the exponentially convex function in information theory, we have the following example.

Example 2.1. [2] *The error function*

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

becomes an exponentially concave function in the form $\text{erf}(\sqrt{x})$, $x \geq 0$, which describes the bit/symbol error probability of communication systems depending on the square root of the underlying signal-to-noise ratio. This shows that the exponentially concave functions can play important part in communication theory and information theory.

We now introduce the concept of the strongly exponentially convex functions, which is the main motivation of this paper.

Definition 2.5. *The function F on the convex set K is said to be strongly exponentially convex with respect to an arbitrary non-negative bifunction $G(\cdot, \cdot)$, if there exists a constant $\mu > 0$, such that*

$$e^{F(u+t(v-u))} \leq (1-t)e^F(u) + te^F(v) - \mu t(1-t)G(v, u), \forall u, v \in K, t \in [0, 1]. \quad (3)$$

The function F is said to be strongly exponentially concave, if and only if, $-F$ is strongly exponentially convex.

If $t = \frac{1}{2}$ and $\mu = 1$, then

$$e^{F(\frac{u+v}{2})} \leq \frac{e^F(u) + e^F(v)}{2} - \frac{1}{4}G(v, u), \quad \forall u, v \in K, t \in [0, 1]. \quad (4)$$

The function F is called the strongly exponentially J -convex function.

We also introduce the concept of strongly exponentially affine convex functions.

Definition 2.6. *A positive function F on the convex set K is said to be strongly affine exponentially convex with respect to an arbitrary non-negative bifunction $G(\cdot, \cdot)$, if there exists a constant $\mu > 0$, such that*

$$e^{F(u+t(v-u))} = (1-t)e^{F(u)} + te^{F(v)} - \mu t(1-t)G(v, u), \quad \forall u, v \in K, t \in [0, 1]. \quad (5)$$

Also, we say that the positive function F is strongly exponentially affine J -convex function, if

$$e^{F(\frac{u+v}{2})} = \frac{e^{F(u)} + e^{F(v)}}{2} - \frac{1}{4}\mu G(v, u), \quad \forall u, v \in K, t \in [0, 1]. \quad (6)$$

We now discuss some special cases, which appears to be new ones.

I. If $G(v, u) = \|v - u\|^\sigma$, $\sigma > 1$, then the strongly exponentially convex functions reduces to:

$$e^{F(u+t(v-u))} \leq (1-t)e^{F(u)} + te^{F(v)} - \mu t(1-t)\|v - u\|^2, \quad \forall u, v \in K, t \in [0, 1],$$

which is called the higher order strongly exponentially convex functions. It is itself an interesting problem to study its characterizations and applications in various field of pure and applied sciences.

II. If $G(v, u) = G(v - u)$ then strongly exponentially convex function becomes:

$$e^{F(u+t(v-u))} \leq (1-t)e^{F(u)} + te^{F(v)} - \mu t(1-t)G(v - u), \quad \forall u, v \in K, t \in [0, 1].$$

For the properties of the strongly convex functions in optimization, inequalities and equilibrium problems, see [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] and the references therein..

Definition 2.7. *The function F on the convex set K is said to be strongly exponentially quasi convex with respect to an arbitrary non-negative bifunction $G(.,.)$, if there exists a constant $\mu > 0$ such that*

$$e^{F(u+t(v-u))} \leq \max\{e^{F(u)}, e^{F(v)}\} - \mu t(1-t)G(v, u), \quad \forall u, v \in K, t \in [0, 1].$$

Definition 2.8. *A positive function F on the convex set K is said to be strongly exponentially log-convex with respect to the bifunction $G(v, u)$, if there exist a constant $\mu > 0$ such that*

$$e^{F(u+t(v-u))} \leq e^{(F(u))^{1-t}} e^{(F(v))^t} - \mu t(1-t)G(v, u), \quad \forall u, v \in K, t \in [0, 1],$$

where $F(\cdot) > 0$.

From this Definition, we have

$$\begin{aligned} e^{F(u+t(v-u))} &\leq e^{(F(u))^{1-t}} e^{(F(v))^t} - \mu t(1-t)G(v, u), \\ &= (1-t)e^{F(u)} + te^{F(v)} - \mu t(1-t)G(v, u) \end{aligned}$$

This shows that every strongly exponentially lg-convex function is a strongly exponentially convex function, but the converse is not true.

In fact, we have

$$\begin{aligned} e^{F(u+t(v-u))} &\leq e^{(F(u))^{1-t}} e^{(F(v))^t} - \mu t(1-t)G(v, u) \\ &\leq (1-t)e^{F(u)} + te^{F(v)} - \mu t(1-t)G(v, u) \\ &\leq \max\{e^{F(u)}, e^{F(v)}\} - \mu t(1-t)G(v, u) \end{aligned}$$

This shows that every strongly exponentially log-convex function is a strongly exponentially convex function and every strongly exponentially convex function is a strongly exponentially quasi-convex function. However, the converse is not true.

Definition 2.9. *A differentiable function F on the convex set K is said to be strongly exponentially pseudo G -convex function with respect to an arbitrary bifunction $G(.,.)$, if and only if, if there exists a constant $\mu > 0$ such that*

$$\begin{aligned} \langle e^{F(u)} F'(u), v - u \rangle + \mu G(u, v) &\geq 0 \\ \Rightarrow \\ e^{F(v)} - e^{F(u)} &\geq 0, \quad \forall u, v \in K. \end{aligned}$$

We also need the following assumptions regarding the bifunction $G(.,.)$, which is due to Noor and Noor [21] and plays a crucial part in the derivation of our results.

Condition N. Let $G(.,.)$ satisfy the assumptions

$$\begin{aligned} G(u, u + t(v - u)) &= -t^2 G(v, u) \\ G(v, u + t(v - u)) &= (1 - t)^2 G(v, u), \quad \forall u, v \in K, t \in [0, 1]. \end{aligned}$$

Clearly for $t = 0$, we have $G(u, u) = 0$. Thus, it is clear that $G(u, v) = 0$, if and only if, $u = v, \forall u, v \in K$.

3. Main Results

In this section, we consider some basic properties of generalized strongly convex functions.

Theorem 3.1. *Let F be a differentiable function on the convex set K and let Condition N hold. Then the function F is strongly exponentially convex function with respect to the non-negative bifunction $G(.,.)$, if and only if,*

$$e^{F(v)} - e^{F(u)} \geq \langle e^{F(u)} F'(u), v - u \rangle + \mu G(v, u), \quad \forall v, u \in K. \quad (7)$$

Proof. Let F be a strongly exponentially convex function on the convex set K . Then

$$e^{F(u+t(v-u))} \leq (1-t)e^{F(u)} + te^{F(v)} - t(1-t)\mu G(v, u), \quad \forall u, v \in K,$$

which can be written as

$$e^{F(v)} - e^{F(u)} \geq \left\{ \frac{e^{F(u+t(v-u))} - e^{F(u)}}{t} \right\} + (1-t)\mu G(v, u).$$

Taking the limit in the above inequality as $t \rightarrow 0$, we have

$$e^{F(v)} - e^{F(u)} \geq \langle e^{F(u)} F'(u), v - u \rangle + \mu G(v, u),$$

which is (7), the required result.

Conversely, let (7) hold. Then $\forall u, v \in K, t \in [0, 1]$, $v_t = u + t(v - u) \in K$ and using Condition N, we have

$$\begin{aligned} e^{F(v)} - e^{F(v_t)} &\geq \langle e^{F(v_t)} F'(v_t), v - v_t \rangle + \mu G(v, v_t) \\ &= (1-t) \langle e^{F(v_t)} F'(v_t), v - u \rangle + \mu(1-t)^2 G(v, u). \end{aligned} \quad (8)$$

In a similar way, we have

$$\begin{aligned} e^{F(u)} - e^{F(v_t)} &\geq \langle e^{F(v_t)} F'(v_t), u - v_t \rangle + \mu G(u, v_t) \\ &= -t \langle e^{F(v_t)} F'(v_t), v - u \rangle + \mu t^2 G(v, u). \end{aligned} \quad (9)$$

Multiplying (8) by t and (9) by $(1-t)$ and adding the resultant, we have

$$e^{F(u+t(v-u))} \leq (1-t)e^{F(u)} + te^{F(v)} - t(1-t)\mu G(v, u),$$

showing that F is a strongly exponentially convex function. \square

Theorem 3.2. *Let F be differentiable on the convex set K and let Condition N hold. Then, (7) holds, if and only if,*

$$\langle e^{F(u)} F'(u) - e^{F(v)} F'(v), u - v \rangle \geq \mu \{G(v, u) + G(u, v)\}, \quad \forall u, v \in K, t \in [0, 1]. \quad (10)$$

Proof. Let F be a strongly exponentially convex function on the convex set K . Then, from Theorem 3.1, we have

$$e^{F(v)} - e^{F(u)} \geq \langle e^{F(u)} F'(u), v - u \rangle + \mu G(v, u) \quad \forall u, v \in K. \quad (11)$$

Changing the role of u and v in (11), we have

$$e^{F(u)} - e^{F(v)} \geq \langle e^{F(v)} F'(v), u - v \rangle + \mu G(u, v) \quad \forall u, v \in K. \quad (12)$$

Adding (11) and (12), we have

$$\langle e^{F(u)} F'(u) - e^{F(v)} F'(v), u - v \rangle \geq \mu \{G(v, u) + G(u, v)\},$$

the required (10).

Conversely, let F' satisfy (10). Then from (16), we have

$$\langle e^{F(v)} F'(v), u - v \rangle \leq \langle e^{F(u)} F'(u), u - v \rangle - \mu \{G(v, u) + G(u, v)\}, \quad (13)$$

Since K is an convex set, $\forall u, v \in K, t \in [0, 1]$ $v_t = u + t(v - u) \in K$.

Taking $v = v_t$ in (13) and using Condition N, we have

$$\begin{aligned}\langle e^{F(v_t)} F'(v_t), u - v_t \rangle &\leq \langle e^{F(u)} F'(u), u - v_t \rangle - \mu\{G(v_t, u) + G(u, v_t)\} \\ &= -t\langle e^{F(u)} F'(u), v - u \rangle - 2t^2\mu G(v, u),\end{aligned}$$

which implies that

$$\langle e^{F(v_t)} F'(v_t), v - u \rangle \geq \langle e^{F(u)} F'(u), v - u \rangle + 2t\mu G(v, u). \quad (14)$$

Consider the auxiliary function

$$g(t) = e^{F(u+t(v-u))},$$

from which, we have

$$g(1) = e^{F(v)}, \quad g(0) = e^{F(u)}.$$

Then, from (14), we have

$$\begin{aligned}g'(t) &= \langle e^{F(v_t)} F'(v_t), v - u \rangle \\ &\geq \langle e^{F(u)} F'(u), v - u \rangle + 2\mu t G(v, u).\end{aligned} \quad (15)$$

Integrating (15) between 0 and 1, we have

$$g(1) - g(0) = \int_0^1 g'(t) dt \geq \langle e^{F(u)} F'(u), v - u \rangle + \mu G(v, u).$$

Thus it follows that

$$e^{F(v)} - e^{F(u)} \geq \langle e^{F(u)} F'(u), v - u \rangle + \mu G(v, u),$$

which is the required (7). \square

Theorem 3.1 and Theorem 3.2, enable us to introduce the following new concepts.

Definition 3.1. The differential $F'(\cdot)$ of the strongly exponentially convex functions is said to be strongly exponentially monotone with respect to an arbitrary bifunction $G(\cdot, \cdot)$, if

$$\langle e^{F(u)} F'(u) - e^{F(v)} F'(v), u - v \rangle \geq \mu\{G(v, u) + G(u, v)\}, \forall u, v \in H.$$

Definition 3.2. The differential $F'(\cdot)$ of the exponentially convex functions is said to be exponentially monotone, if

$$\langle e^{F(u)} F'(u) - e^{F(v)} F'(v), u - v \rangle \geq 0, \forall u, v \in H.$$

Definition 3.3. The differential $F'(\cdot)$ of the strongly exponentially convex functions is said to be strongly exponentially G -pseudo monotone with respect to an arbitrary bifunction $G(\cdot, \cdot)$, if

$$\langle e^{F(u)} F'(u), v - u \rangle \geq 0.$$

implies that

$$\langle e^{F(v)} F'(v), v - u \rangle \geq \mu G(u, v), \forall u, v \in H. \quad (16)$$

We now give a necessary condition for strongly exponentially G -pseudo-convex function.

Theorem 3.3. Let F' be strongly exponentially G -pseudomonotone and Condition N hold. Then F is a strongly exponentially G -pseudo-invex function.

Proof. Let F' be a strongly exponentially G -pseudomonotone. Then, $\forall u, v \in K$,

$$\langle e^{F(u)} F'(u), v - u \rangle \geq 0.$$

implies that

$$\langle e^{F(v)} F'(v), v - u \rangle \geq \mu G(u, v). \quad (17)$$

Since K is a convex set, $\forall u, v \in K, t \in [0, 1], v_t = u + t(v - u) \in K$.

Taking $v = v_t$ in (17) and using condition N, we have

$$\langle e^{F(v_t)} F'(v_t), v - u \rangle \geq t\mu G(v, u). \quad (18)$$

Consider the auxiliary function

$$g(t) = e^{F(u+t(v-u))} = e^{F(v_t)}, \quad \forall u, v \in K, t \in [0, 1],$$

which is differentiable, since F is a differentiable function. Then, using (18), we have

$$g'(t) = \langle e^{F(v_t)} F'(v_t), v - u \rangle \geq t\mu G(v, u).$$

Integrating the above relation between 0 to 1, we have

$$g(1) - g(0) = \int_0^1 g'(t) dt \geq \frac{\mu}{2} G(v, u),$$

that is,

$$e^{F(v)} - e^{F(u)} \geq \frac{\mu}{2} G(v, u),$$

showing that F is a strongly exponentially G -pseudo-convex function. \square

Definition 3.4. The function F is said to be sharply strongly exponentially pseudo convex, if there exists a constant $\mu > 0$ such that

$$\begin{aligned} \langle e^{F(u)} F'(u), v - u \rangle &\geq 0 \\ \Rightarrow \\ F(v) &\geq e^{F(v+t(u-v))} + \mu t(1-t)G(v, u) \quad \forall u, v \in K, t \in [0, 1]. \end{aligned}$$

Theorem 3.4. Let F be a sharply strongly exponentially pseudo convex function on K with a constant $\mu > 0$. Then

$$\langle e^{F(v)} F'(v), v - u \rangle \geq \mu G(v, u) \quad \forall u, v \in K.$$

Proof. Let F be a sharply strongly exponentially pseudo convex function on K . Then

$$e^{F(v)} \geq e^{F(v+t(u-v))} + \mu t(1-t)G(v, u), \quad \forall u, v \in K, t \in [0, 1].$$

from which we have

$$\frac{e^{F(v+t(u-v))} - e^{F(v)}}{t} + \mu t(1-t)G(v, u) \leq 0.$$

Taking limit in the above inequality, as $t \rightarrow 0$, we have

$$\langle e^{F(v)} F'(v), v - u \rangle \geq \mu G(v, u),$$

the required result. \square

We now discuss the optimality condition for the differentiable strongly exponentially convex functions, which is the main motivation of our next result.

Theorem 3.5. Let F be a differentiable strongly exponentially convex function with modulus $\mu > 0$. If $u \in K$ is the minimum of the function F , then

$$e^{F(v)} - e^{F(u)} \geq \mu G(v, u), \quad \forall u, v \in K. \quad (19)$$

Proof. Let $u \in K$ be a minimum of the function F . Then

$$F(u) \leq F(v), \forall v \in K$$

from which, we have

$$e^{F(u)} \leq e^{F(v)}, \forall v \in K \quad (20)$$

Since K is a convex set, so, $\forall u, v \in K, \quad t \in [0, 1]$,

$$v_t = (1 - t)u + tv \in K.$$

Taking $v = v_t$ in (20), we have

$$\begin{aligned} 0 &\leq \lim_{t \rightarrow 0} \left\{ \frac{e^{F(u+t(v-u))} - e^{F(u)}}{t} \right\} \\ &= \langle e^{F(u)} F'(u), v - u \rangle. \end{aligned} \quad (21)$$

Since F is differentiable strongly exponentially convex function, so

$$\begin{aligned} e^{F(u+t(v-u))} &\leq e^{F(u)} + t(e^{F(v)} - e^{F(u)}) \\ &\quad - \mu t(1 - t)G(v, u), \quad u, v \in K, t \in [0, 1], \end{aligned}$$

from which, using (21), we have

$$\begin{aligned} e^{F(v)} - e^{F(u)} &\geq \lim_{t \rightarrow 0} \frac{e^{F(u+t(v-u))} - e^{F(u)}}{t} + \mu G(v, u). \\ &= \langle e^{F(u)} F'(u), v - u \rangle + \mu G(v, u) \\ &\geq \mu G(v, u), \end{aligned}$$

the required result (19). □

Remark 3.1. We would like to mention that, if

$$\langle e^{F(u)} F'(u), v - u \rangle + \mu G(v, u) \geq 0, \quad \forall u, v \in K,$$

then $u \in K$ is the minimum of the function F .

We would like to emphasize that the minimum $u \in K$ of the exponentially convex functions can be characterized of the inequality

$$\langle e^{F(u)} F'(u), v - u \rangle \geq 0, \forall v \in K, \quad (22)$$

which is called the exponential variational inequality, which appears to be new on. It is an interesting problem to study the existence of a unique solution of the inequality (22) and its applications.

Definition 3.5. A function F is said to be a exponentially pseudo convex function, if there exists a strictly positive bifunction $b(., .)$, such that

$$\begin{aligned} e^{F(v)} &< e^{F(u)} \\ \Rightarrow \\ e^{F(u+t(v-u))} &< e^{F(u)} + t(t-1)b(v, u), \forall u, v \in K, t \in [0, 1]. \end{aligned}$$

Theorem 3.6. If the function F is exponentially convex function such that $e^{F(v)} < e^{F(u)}$, then the function F is strongly exponentially pseudo convex.

Proof. Since $e^{F(v)} < e^{F(u)}$ and F is strongly exponentially convex function, then $\forall u, v \in K$, $t \in [0, 1]$, we have

$$\begin{aligned} e^{F(u+t(v-u))} &\leq e^{F(u)} + t(e^{F(v)} - e^{F(u)}) - \mu t(1-t)G(v, u) \\ &< e^{F(u)} + t(1-t)(e^{F(v)} - e^{F(u)}) - \mu t(1-t)G(v, u) \\ &= e^{F(u)} + t(t-1)(e^{F(u)} - e^{F(v)}) - \mu t(1-t)G(v, u) \\ &< e^{F(u)} + t(t-1)b(u, v) - \mu t(1-t)G(v, u), \end{aligned}$$

where $b(u, v) = e^{F(u)} - e^{F(v)} > 0$, the required result. This shows that the function F is strongly exponentially convex function. \square

It is well known that each strongly convex functions is of the form $f \pm \|\cdot\|^2$, where f is a convex function. Similar result is proved for the strongly exponentially convex functions. In this direction, we have:

Theorem 3.7. *Let f be a strongly exponentially affine function with respect to an arbitrary bifunction $G(\cdot, \cdot)$. Then F is a strongly exponentially convex function with respect to the same arbitrary bifunction $G(\cdot, \cdot)$, if and only if, $g = F - f$ is a exponentially convex function.*

Proof. Let f be strongly exponentially affine function with respect to the arbitrary bifunction $G(\cdot, \cdot)$. Then

$$e^{f((1-t)u+tv)} = (1-t)e^{f(u)} + te^{f(v)} - \mu t(1-t)G(v, u). \quad (23)$$

From the strongly exponentially convexity of F , we have

$$e^{F((1-t)u+tv)} \leq (1-t)e^{F(u)} + te^{F(v)} - \mu t(1-t)G(v, u). \quad (24)$$

From (23) and (24), we have

$$e^{F((1-t)u+tv)} - e^{f((1-t)u+tv)} \leq (1-t)(e^{F(u)} - e^{f(u)}) + t(e^{F(v)} - e^{f(v)}), \quad (25)$$

from which it follows that

$$\begin{aligned} e^{g((1-t)u+tv)} &= e^{F((1-t)u+tv)} - e^{f((1-t)u+tv)} \\ &\leq (1-t)(e^{F(u)} - e^{f(u)}) + t(e^{F(v)} - e^{f(v)}), \end{aligned}$$

which show that $g = F - f$ is an exponentially convex function.

The inverse implication is obvious. \square

We would like to remark that one can show that a function F is a strongly exponentially convex function, if and only if, F is strongly exponentially affine function essentially using the technique of Adamek [1] and Noor et al. [21].

Conclusion

In this paper, we have introduced and studied a new class of convex functions with respect to any arbitrary bifunction, which is called the strongly exponentially convex function. It is shown that several new classes of strongly exponentially convex functions can be obtained as special cases of these strongly exponentially convex functions. We have studied the basic properties of these functions. Several new and interesting results have been obtained. It is shown that the optimality conditions of the differentiable exponentially convex functions can be characterized by a class of variational inequality, which is called the exponentially variational inequality. The qualitative study of the exponentially variational inequalities is an interesting problem for future research. It is expected that the ideas and techniques of this paper may stimulate further research.

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