# QUANTITATIVE ESTIMATES FOR A NEW COMPLEX Q-DURRMEYER TYPE OPERATORS ON COMPACT DISKS

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*In the present article, the upper bound and Voronovskaya type result with quantitative estimate and the exact degree of approximation for a new complex q-Bernstein-Durrmeyer operators attached to analytic functions on compact disks are obtained. In this way, we put in evidence the over convergence phenomenon for the q-Bernstein-Durrmeyer polynomials, namely the extensions of approximation properties (with quantitative estimates) from real intervals to compact disks in the complex plane.*

**Keywords**: q-Durrmeyer type operators, q-integers, complex approximation, Voronovskaja-type result, exact degree of approximation.

## 1. Introduction

Since last few years, the study of linear positive operators defined on a complex domain has been an active area of research in approximation theory. S. N. Bernstein [19] was the first one who initiated complex approximation and introduced complex Bernstein polynomials by

$$B_n(f;z) = \sum_{k=0}^{n} \binom{n}{k} z^k (1-z)^{n-k} f(k/n).$$

If $f\colon \mathrm{G} \to \mathbb{C}$ is an analytic function in the open set $\mathrm{G} \subset \mathbb{C}$, with $\overline{D}_1 \subset \mathrm{G}$ (where $D_1 = \{z \in \mathbb{C}: |z| > 1\}$, then Bernstein proved that the complex Bernstein polynomials converges to $f$ uniformly in $\overline{D}_1$.

Sorin Gal pioneered the study of the upper quantitative estimates for the uniform convergence of $B_n(f)$ to $f$ in [7]. In [9], Gal proved the Voronovskaja type results with quantitative estimates for the complex Bernstein polynomials. Anastassiou-Gal [6], Gal [8], Gal-Gupta [13], and Gupta [15] established quantitative estimates for certain other variants of Durrmeyer type operators.

In 2011, Mahmudov [20] obtained the order of simultaneous approximation and Voronovskaja type theorems with quantitative esimates for the

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complex one parameter class of Bernstein-Durrmeyer polynomials on compact disks. In the present paper, we establish the exact order in ordinary approximation and a Voronovskaja type theorem with a quantitative esimate for the complex modified Bernstein-Schurer operators based on q-integers on compact disks.

Recently, Agarwal-Gupta [4] studied the upper quantitative estimates for q-Bernstein-Durrmeyer operators on compact disks. To make the convergence faster, Ren and Zeng used the King type approach for these operators on compact disks in [24]. A q-analogue of genuine Bernstein-Durrmeyer operators on compact disks is given in [21]. Also Agrawal-Sathish has studied the over convergence properties of new type of q-Bernstein Schurer operators on compact disks in [5]. Recently, Gal [14] and Gupta-Agarwal [16] studied the over convergence properties for several integral operators. In 2016, Gal and Gupta [10] studied the approximation properties of the complex version of Durrmeyer type operators based on Polya distribution, attached to analytic functions on a disk. In the recent years several researchers have studied in this direction for different sequences of linear positive operators (see [1-3], [11-12], [17-18], [22-23], [25-26], etc.).

Let $p \in N^0 := N \cup \{0\}$ (the set of all non-negative integers) and $0 < q < 1$. For $f \in C[0,1+p]$ the q-modified complex Bernstein-Schurer operators are defined for $\forall z \in \mathbb{C}$, as

$$D_{n,p}(f;q,z) = \frac{[n+p+1]_q[n]_q}{[n+p]_q}\sum_{k=0}^{n+p} b_{n+p,k}(q;z)q^{-k}\int_0^{\frac{[n+p]_q}{[n]_q}} f(t)b_{n+p,k}(q;qt)d_q(t), \qquad (1)$$

where

$$b_{n+p,k}(q;z) = \frac{[n]_q}{[n+p]_q}\binom{n+p}{k} z^k \left(\frac{[n+p]_q}{[n]_q} - z\right)_q^{n+p-k}.$$

Let $D_R$ be the disk $D_R := \{z \in \mathbb{C}: |z| < R\}$ in the complex plane $\mathbb{C}$. Let us denote by $H(D_R)$, the space of all analytic functions on $D_R$. For $f \in H(D_R)$, we may write

$$f(z) = \sum_{m=0}^{\infty} c_m z^m$$

In this paper, we have shown the overconvergence phenomenon for a new type of $q$-Bernstein-Durrmeyer type operators, namely the extensions of the approximation properties with quantitative estimates from real intervals to complex domain.

## 2. Basic Results

In the sequel, we require the following results.

**Lemma 2.1.** *Let* $\boldsymbol{D_{n,p}(f;q,z)}$ *be as defined in* $(\mathbf{1})$ *and* $\mathbf{0}<\boldsymbol{q}<\mathbf{1}$*. Then,* $\boldsymbol{D_{n,p}(t^m;q,z)}$ *is a polynomial in* $\boldsymbol{z}$ *of degree* $\leq \boldsymbol{min\{m,n\}}$ *and*

$$\boldsymbol{D_{n,p}(t^m;q,z)} = \frac{[\boldsymbol{n}+\boldsymbol{p}+\mathbf{1}]_q!}{[\boldsymbol{n}+\boldsymbol{p}+\boldsymbol{m}+\mathbf{1}]_q!}\frac{[\boldsymbol{n}+\boldsymbol{p}+\mathbf{1}]_q^m}{[\boldsymbol{n}]_q^m}\sum_{s=0}^{m}\boldsymbol{c_s(m,q)[n]_q^s B_{n,p}(e_s,q,z)}, \qquad (\mathbf{2})$$

*where* $c_s(m,q)>0$ *are certain constants depending on* $m$ *and* $q$ *and* $B_{n,p}(f,q,z)$ *is the q-Bernstein-Schurer polynomial defined by* $\sum_{k=0}^{n+p} b_{n+p,k}(q;z)f\left([k]_q/[n]_q\right)$*.*

**Proof.** By simple computations and using $[k+s]_q=[s]_q+q^s[k]_q$, the proof of the lemma easily follows hence the details are omitted.

**Lemma 2.2.** *Let* $0<q<1$*. Then, for all* $m,n\in N$ *and* $p\in N^0$ *such that* $m\leq n+p$*, we have*

$$\frac{[n+p+1]_q!}{[n+p+m+1]_q!}\sum_{s=0}^{m}c_s(m,q)[n+p]_q^s\leq 1.$$

**Proof.** In view of Lemma 2.1 with $e_m(t)=t^m$, we obtain

$$\boldsymbol{D_{n,p}\left(t^m;q,\frac{[n+p]_q}{[n]_q}\right)} = \frac{[\boldsymbol{n}+\boldsymbol{p}+\mathbf{1}]_q!}{[\boldsymbol{n}+\boldsymbol{p}+\boldsymbol{m}+\mathbf{1}]_q!}\frac{[\boldsymbol{n}+\boldsymbol{p}]_q^m}{[\boldsymbol{n}]_q^m}\sum_{s=0}^{m}\boldsymbol{c_s(m,q)[n]_q^s B_{n,p}\left(e_s;q,\frac{[n+p]_q}{[n]_q}\right)}.$$

If we consider the operators

$$B_{n,p}(e_s;q,z)=\frac{[n]_q^{n+p}}{[n+p]_q^{n+p}}\sum_{k=0}^{n+p}\binom{n+p}{k}_q z^k\left(\frac{[n+p]_q}{[n]_q}-z\right)_q^{n+p-k}\left(\frac{[k]_q}{[n]_q}\right)^s$$

And putting $z=\frac{[n+p]_q}{[n]_q}w$, then we get

$$\frac{[n]_q^{n+p}}{[n+p]_q^{n+p}}\sum_{k=0}^{n+p}\binom{n+p}{k}_q z^k\left(\frac{[n+p]_q}{[n]_q}-z\right)_q^{n+p-k}\left(\frac{[k]_q}{[n]_q}\right)^s$$

$$= \sum_{k=0}^{n+p} \binom{n+p}{k}_q w^k (1-w)_q^{n+p-k} \left(\frac{[k]_q}{[n]_q}\right)^s$$
$$= \frac{1}{[n]_q^s} \sum_{k=0}^{n+p} \binom{n+p}{k}_q w^k (1-w)_q^{n+p-k} [k]_q^s$$

By ([7], p.61, Theorem 1.5.6), since $\sum_{k=0}^{n} \binom{n}{k}_q w^k (1-w)_q^{n-k} \left(\frac{[k]_q}{[n]_q}\right)^s = 1$ at $w = 1$ for all $s = 0,1,2$ ...therefore,

$$\frac{[n]_q^{n+p}}{[n+p]_q^{n+p}} \sum_{k=0}^{n+p} \binom{n+p}{k}_q z^k \left(\frac{[n+p]_q}{[n]_q} - z\right)_q^{n+p-k} \left(\frac{[k]_q}{[n]_q}\right)^s = \frac{[n+p]_q^s}{[n]_q^s}$$

at $z = \frac{[n+p]_q}{[n]_q}$. Further

$$\boldsymbol{D_{n,p}\left(t^m; q, \frac{[n+p]_q}{[n]_q}\right) = \frac{[n+p+1]_q!}{[n+p+m+1]_q!} \frac{[n+p]_q^m}{[n]_q^m} \sum_{s=0}^{m} c_s(m,q)[n+p]_q^s.} \quad (3)$$

Since $\boldsymbol{b_{n+p,k}\left(q; \frac{[n+p]_q}{[n]_q}\right) = 0}$ for $\boldsymbol{k = 0,1,2,\dots,n+p-1}$ and $\boldsymbol{b_{n+p,k}\left(q; \frac{[n+p]_q}{[n]_q}\right) = 1}$ for $\boldsymbol{k = n+p}$ we have

$$\boldsymbol{D_{n,p}\left(t^m; q, \frac{[n+p]_q}{[n]_q}\right) = \frac{[n+p+1]_q}{[n+p]_q} q^{-(n+p)} b_{n+p,n+p}\left(q; \frac{[n+p]_q}{[n]_q}\right) \times \int_0^{\frac{[n+p]_q}{[n]_q}} t^m b_{n+p,n+p}(q; qt) d_q(t)}$$
$$\boldsymbol{= \frac{[n+p+1]_q}{[n+p+m+1]_q} \frac{[n+p]_q^m}{[n]_q^m} \le \frac{[n+p]_q^m}{[n]_q^m}.} \quad (4)$$

From (3) and (4), we obtain

$$\frac{[n+p+1]_q!}{[n+p+m+1]_q!}\sum_{s=0}^{m} c_s(m,q)[n+p]_q^s \leq 1.$$

which completes the proof.

**Lemma 2.3.** Let $r > \frac{[n+p]_q}{[n]_q}$ and $0 < q < 1$. Then, for all $m, n \in N^0$ and $|z| \leq r$, we have $\left|\mathrm{D}_{\mathrm{n,p}}(e_m; \mathrm{q}, \mathrm{z})\right| \leq r^m$.

**Proof.** From Gal ([7] p.61, proof of Theorem 1.5.6), we have $\left|B_{n,q}(e_s; z)\right| \leq r^s$ whenever $|z| \leq r$ and $r \geq 1$. Hence, in view of Lemmas 2.1 and 2.2, for all $m, n \in N^0$ and $|z| \leq r$, $r \geq \frac{[n+p]_q}{[n]_q}$ we get

$$\left|\boldsymbol{D_{n,p}(e_m; q, z)}\right| \leq \frac{[\boldsymbol{n}+\boldsymbol{p}+\boldsymbol{1}]_{\boldsymbol{q}}!}{[\boldsymbol{n}+\boldsymbol{p}+\boldsymbol{m}+\boldsymbol{1}]_{\boldsymbol{q}}!}\frac{[\boldsymbol{n}+\boldsymbol{p}]_{\boldsymbol{q}}^{\boldsymbol{m}}}{[\boldsymbol{n}]_{\boldsymbol{q}}^{\boldsymbol{m}}}\sum_{s=0}^{m} \boldsymbol{c_s(m,q)}[\boldsymbol{n}]_{\boldsymbol{q}}^{\boldsymbol{s}}\left|\boldsymbol{B_{n,p}(e_s; q, z)}\right|$$

$$\leq \frac{[\boldsymbol{n}+\boldsymbol{p}+\boldsymbol{1}]_{\boldsymbol{q}}!}{[\boldsymbol{n}+\boldsymbol{p}+\boldsymbol{m}+\boldsymbol{1}]_{\boldsymbol{q}}!}\frac{[\boldsymbol{n}+\boldsymbol{p}]_{\boldsymbol{q}}^{\boldsymbol{m}}}{[\boldsymbol{n}]_{\boldsymbol{q}}^{\boldsymbol{m}}}\sum_{s=0}^{m} \boldsymbol{c_s(m,q)}\frac{[\boldsymbol{n}+\boldsymbol{p}]_{\boldsymbol{q}}^{\boldsymbol{s}}}{[\boldsymbol{n}]_{\boldsymbol{q}}^{\boldsymbol{s}}}\left(\mathbf{r}\frac{[\boldsymbol{n}]_{\boldsymbol{q}}}{[\boldsymbol{n}+\boldsymbol{p}]_{\boldsymbol{q}}}\right)^s$$

$$\leq \frac{[\boldsymbol{n}+\boldsymbol{p}+\boldsymbol{1}]_{\boldsymbol{q}}!}{[\boldsymbol{n}+\boldsymbol{p}+\boldsymbol{m}+\boldsymbol{1}]_{\boldsymbol{q}}!}\mathbf{r}^{\mathbf{m}}\sum_{s=0}^{m} \boldsymbol{c_s(m,q)}[\boldsymbol{n}+\boldsymbol{p}]_{\boldsymbol{q}}^{\boldsymbol{s}} \leq \boldsymbol{r^m}.$$

**Remark2.1.** By simple computations, we have

$$\boldsymbol{z}\left(\frac{[\boldsymbol{n}+\boldsymbol{p}]_{\boldsymbol{q}}}{[\boldsymbol{n}]_{\boldsymbol{q}}}-\boldsymbol{z}\right)\boldsymbol{D_q}\left(\boldsymbol{b_{n+p,k}(q;z)}\right)$$

$$= \boldsymbol{b_{n+p,k}(q;z)}\left([\boldsymbol{k}]_{\boldsymbol{q}}\frac{[\boldsymbol{n}+\boldsymbol{p}]_{\boldsymbol{q}}}{[\boldsymbol{n}]_{\boldsymbol{q}}}-[\boldsymbol{n}+\boldsymbol{p}]_{\boldsymbol{q}}\boldsymbol{z}\right)$$

and

$$\boldsymbol{z}\left(\frac{[\boldsymbol{n}+\boldsymbol{p}]_{\boldsymbol{q}}}{[\boldsymbol{n}]_{\boldsymbol{q}}}-\boldsymbol{z}\right)\boldsymbol{D_q b_{n+p,k}(q;z)} = \boldsymbol{b_{n+p,k}(q;z)}\left(\frac{[\boldsymbol{k}]_{\boldsymbol{q}}[\boldsymbol{n}+\boldsymbol{p}]_{\boldsymbol{q}}}{[\boldsymbol{n}]_{\boldsymbol{q}}}-\boldsymbol{qt}[\boldsymbol{n}+\boldsymbol{p}]_{\boldsymbol{q}}\right).$$

**Lemma 2.4.** Let $\boldsymbol{0 < q < 1}$. For all $\boldsymbol{e_m(t) = t^m}$, $\boldsymbol{m \in N^0}$ and $\boldsymbol{z} \in \mathbb{C}$ we have

$$z\left(\frac{[n+p]_q}{[n]_q}-z\right)D_q\left(D_{n,p}(e_m;q,z)\right)$$
$$=\left(q[n+p]_q+[m+2]_q q^{-(m+1)}\right)D_{n,p}(e_{m+1};q,z)$$
$$-\left([m+1]_q q^{-(m+1)}\frac{[n+p]_q}{[n]_q}+[n+p]_q z\right).$$

**Proof.** Applying Remark 2.1, we have

$$z\left(\frac{[n+p]_q}{[n]_q}-z\right)D_q\left(D_{n,p}(e_m;q,z)\right)$$
$$=\frac{[n+p+1]_q[n]_q}{[n+p]_q}\sum_{k=0}^{n+p}z\left(\frac{[n+p]_q}{[n]_q}\right.$$
$$\left.-z\right)_q D_q\left(b_{n+p,k}(q;z)\right)q^{-k}\int_0^{\frac{[n+p]_q}{[n]_q}}t^m b_{n+p,k}(q;qt)d_q(t)$$

$$=\frac{[n+p+1]_q[n]_q}{[n+p]_q}\sum_{k=0}^{n+p}b_{n+p,k}(q;z)\left([k]_q\frac{[n+p]_q}{[n]_q}\right.$$
$$\left.-[n+p]_q z\right)q^{-k}\int_0^{\frac{[n+p]_q}{[n]_q}}t^m b_{n+p,k}(q;qt)d_q(t)$$

$$=\frac{[n+p+1]_q[n]_q}{[n+p]_q}\sum_{k=0}^{n+p}b_{n+p,k}(q;z)\left([k]_q\frac{[n+p]_q}{[n]_q}-[n+p]_q qt+[n+p]_q z\right.$$
$$\left.-[n+p]_q z\right)q^{-k}\int_0^{\frac{[n+p]_q}{[n]_q}}t^m b_{n+p,k}(q;qt)d_q(t)$$

$$=\frac{[n+p+1]_q[n]_q}{[n+p]_q}\sum_{k=0}^{n+p}b_{n+p,k}(q;z)\left([k]_q\frac{[n+p]_q}{[n]_q}\right.$$
$$\left.-[n+p]_q qt\right)q^{-k}\int_0^{\frac{[n+p]_q}{[n]_q}}t^m b_{n+p,k}(q;qt)d_q(t)$$
$$+q[n+p]_q D_{n,p}(e_{m+1};q,z)$$
$$-[n+p]_q z D_{n,p}(e_m;q,z) \qquad (5)$$

Let $\delta(t)=\left(\frac{[n+p]_q}{[n]_q}-t\right)\left(\frac{t}{q}\right)^{m+1}$, then $\delta(qt)=\left(\frac{[n+p]_q}{[n]_q}-qt\right)t^{m+1}$. Using $q$-integration by parts, we have

$$\int_0^{\frac{[n+p]_q}{[n]_q}} D_q\left(b_{n+p,k}(q;qt)\right)\delta(qt)d_q(t)$$

$$=\delta(t)b_{n+p,k}(q;qt)\Big|_0^{\frac{[n+p]_q}{[n]_q}}-\int_0^{\frac{[n+p]_q}{[n]_q}} b_{n+p,k}(q;qt)D_q\big(\delta(t)\big)d_q(t)$$

$$=-\frac{1}{q^{m+1}}\int_0^{\frac{[n+p]_q}{[n]_q}} b_{n+p,k}(q;qt)D_q\left(\frac{[n+p]_q}{[n]_q}t^{m+1}-t^{m+2}\right)d_q(t)$$

$$=-\frac{1}{q^{m+1}}\int_0^{\frac{[n+p]_q}{[n]_q}} b_{n+p,k}(q;qt)\left(\frac{[n+p]_q}{[n]_q}[m+1]_q t^m-[m+2]_q t^{m+1}\right)d_q(t).$$

In view of (5), we get the desired recurrence relation.

## 3. Main Results

Let $P_n(z)$ be a polynomial of degree $n$ of complex variable $z$ with derivative $P_n{}'(z)$. Then, by the Bernstein inequality and the complex mean value theorem, we have

$$\left|D_q\big(P_n(z)\big)\right|\leq\|P_n{}'\|_r\leq\frac{n}{r}\|P_n\|_r, \text{ for all } |z|\leq r, \qquad (6)$$

where$\|.\|_r$denotes the sup-norm on $|z|\leq r$.

Our first main result is the following upper estimate.

**Theorem 3.1.** Let $0<q<1$, $f(z)=\sum_{m=0}^{\infty}c_m z^m$, for all $|z|\leq R$ and let $\frac{[n+p]_q}{[n]_q}\leq r\leq R$. Then, for all $|z|\leq r$ and $n\in N$, we have

$$\left|D_{n,p}(f;q,z)-f(z)\right|\leq\frac{[n+p]_q}{[n]_q}\frac{C_r(f)}{[n+p+1]_q},$$

where $C_r(f) = (1+r)\sum_{m=1}^{\infty}|c_m| m(m+1)\, r^{m-1} < \infty$.

**Proof.** First we show that $D_{n,p}(f;q,z) = \sum_{m=0}^{\infty} C_m D_{n,p}(e_m;q,z)$, where $e_m(z) = z^m, m = 0,1,2,....$ Indeed, denoting $f_m(z) = \sum_{j=0}^{m} c_j z^j$, $|z| \le r$, $m \in N$, by applying the linearity of $D_{n,p}$, we get $D_{n,p}(f_m;q,z) = \sum_{j=0}^{m} c_j D_{n,p}(e_j;q,z)$. For any fixed $n \in N$ and $|z| \le r$ with $r \ge \frac{[n+p]_q}{[n]_q}$, it is enough to show that

$$\lim_{m\to\infty} D_{n,p}(f_m;q,z) = D_{n,p}(f;q,z).$$

But this is immediate from $\lim_{m\to\infty} \|f_m - f\|_r = 0$ and from the inequality

$$\left|D_{n,p}(f_m;q,z) - f(z)\right| \le \frac{[n+p+1]_q[n]_q}{[n+p]_q}\sum_{k=0}^{n+p}\left|b_{n+p,k}(q;z)\right| q^{-k}\int_0^{\frac{[n+p]_q}{[n]_q}} b_{n+p,k}(q;qt)|f_m(t) - f(t)|d_q(t) \le M_{r,n,p}\|f_m - f\|_r$$

valid for all $|z| \le r$, where

$$M_{r,n,p} = \frac{[n+p+1]_q[n]_q}{[n+p]_q}\sum_{k=0}^{n+p}\binom{n+p}{k}_q r^k\left(\frac{[n+p]_q}{[n]_q} + r\right)_q^{n+p-k} q^{-k}\int_0^{\frac{[n+p]_q}{[n]_q}} b_{n+p,k}(q;qt)d_q(t)$$

$$= \sum_{k=0}^{n+p}\binom{n+p}{k}_q r^k\left(\frac{[n+p]_q}{[n]_q} + r\right)_q^{n+p-k}.$$

Therefore, we get

$$\left|D_{n,p}(f;q,z) - f(z)\right| \le \sum_{m=0}^{\infty}|C_m|\left|D_{n,p}(e_m;q,z) - e_m(z)\right|$$
$$= \sum_{m=1}^{\infty}|C_m|\left|D_{n,p}(e_m;q,z) - e_m(z)\right|$$

since $\boldsymbol{D_{n,p}(e_0;q,z)=e_0(z)=1}$. Now from Lemma 2.4, for all $\boldsymbol{m\in N}$ we have

$$\boldsymbol{D_{n,p}(e_m;q,z)-e_m(z)}$$
$$\boldsymbol{=\frac{q^m z\left(\frac{[n+p]_q}{[n]_q}-z\right)}{[n+p+m+1]_q}D_qD_{n,p}(e_{m-1};q,z)}$$
$$\boldsymbol{+\frac{[m]_q\frac{[n+p]_q}{[n]_q}+q^m z[n+p]_q}{[n+p+m+1]_q}\left(D_{n,p}(e_{m-1};q,z)-e_{m-1}(z)\right)}$$
$$\boldsymbol{+\frac{[m]_q\frac{[n+p]_q}{[n]_q}+q^m z[n+p]_q}{[n+p+m+1]_q}z^{m-1}-z^m}$$

$$\boldsymbol{=\frac{q^m z\left(\frac{[n+p]_q}{[n]_q}-z\right)}{[n+p+m+1]_q}D_qD_{n,p}(e_{m-1};q,z)}$$
$$\boldsymbol{+\frac{[m]_q\frac{[n+p]_q}{[n]_q}+q^m z[n+p]_q}{[n+p+m+1]_q}\left(D_{n,p}(e_{m-1};q,z)-e_{m-1}(z)\right)}$$
$$\boldsymbol{+\frac{[m]_q\frac{[n+p]_q}{[n]_q}}{[n+p+m+1]_q}z^{m-1}}$$
$$\boldsymbol{+\frac{\left(q^m[n+p]_q-[n+p+m+1]_q\right)}{[n+p+m+1]_q}z^m}.$$

Since $q^m[n+p]_q-[n+p+m+1]_q=-[m]_q-q^{n+p+m}$, we have

$$\left|D_{n,p}(e_m;q,z)-e_m(z)\right|$$
$$=\frac{q^m r\left(\frac{[n+p]_q}{[n]_q}+r\right)}{[n+p+m+1]_q}\left|D_qD_{n,p}(e_{m-1};q,z)\right|$$
$$+\frac{[m]_q\frac{[n+p]_q}{[n]_q}+q^m r[n+p]_q}{[n+p+m+1]_q}\left|D_{n,p}(e_{m-1};q,z)-e_{m-1}(z)\right|$$
$$+\frac{[m]_q\frac{[n+p]_q}{[n]_q}}{[n+p+m+1]_q}r^{m-1}$$
$$+\frac{\left(q^m[n+p]_q-[n+p+m+1]_q\right)}{[n+p+m+1]_q}r^m.$$

Using (6) and Lemma 2.3 we obtain

$$\frac{q^m r\left(\frac{[n+p]_q}{[n]_q}+r\right)}{[n+p+m+1]_q}\left|D_q D_{n,p}(e_{m-1};q,z)\right|$$
$$\leq \frac{q^m r\left(\frac{[n+p]_q}{[n]_q}+r\right)(m-1)}{[n+p+m+1]_q r}\left\|D_{n,p}(e_{m-1};q,.)\right\|_r$$
$$\leq \frac{q^m\left(\frac{[n+p]_q}{[n]_q}+r\right)(m-1)}{[n+p+m+1]_q}\left\|D_{n,p}(e_{m-1};q,.)\right\|_r$$
$$\leq \frac{\frac{[n+p]_q}{[n]_q}(1+r)}{[n+p+1]_q}r^{m-1}.$$

Also, we have

$$\frac{[m]_q\frac{[n+p]_q}{[n]_q}}{[n+p+m+1]_q}r^{m-1}\leq \frac{[m+1]_q\frac{[n+p]_q}{[n]_q}}{[n+p+1]_q}r^{m-1}$$

and

$$\frac{[m]_q+q^{n+p+m}}{[n+p+m+1]_q}r^m\leq \frac{[m+1]_q+q^{n+p+m}}{[n+p+1]_q}r^m.$$

From $[n+p+m+1]_q=q^m[n+p]_q+[m]_q+q^{n+p+m}$, it follows that

$$\frac{q^m[n+p]_q r+[m]_q\frac{[n+p]_q}{[n]_q}}{[n+p+m+1]_q}\leq r.$$

Hence, we get

$$\left|D_{n,p}(e_m;q,z)-e_m(z)\right|$$
$$\leq \frac{\frac{[n+p]_q}{[n]_q}(1+r)(m-1)}{[n+p+1]_q}r^{m-1}+r\left|D_{n,p}(e_{m-1};q,z)-e_{m-1}(z)\right|$$
$$+\frac{[m+1]_q\frac{[n+p]_q}{[n]_q}}{[n+p+1]_q}r^{m-1}+\frac{[m+1]_q\frac{[n+p]_q}{[n]_q}}{[n+p+1]_q}r^m$$

$$\leq \frac{\frac{[n+p]_q}{[n]_q}(1+r)(m-1)}{[n+p+1]_q} r^{m-1} + r\left|D_{n,p}(e_{m-1};q,z) - e_{m-1}(z)\right|$$
$$+ \frac{[m+1]_q \frac{[n+p]_q}{[n]_q}}{[n+p+1]_q} r^{m-1}(1+r)$$
$$\leq 2m \frac{\frac{[n+p]_q}{[n]_q}(1+r)}{[n+p+1]_q} r^{m-1} + r\left|D_{n,p}(e_{m-1};q,z) - e_{m-1}(z)\right|.$$

By writing the last inequality for $m = 2,3,\ldots$ we can easily obtain step by step, the following

$$\left|D_{n,p}(e_m;q,z) - e_m(z)\right|$$
$$\leq r\left( r\left|D_{n,p}(e_{m-2};q,z) - e_{m-2}(z)\right| + \frac{2(m-1)\frac{[n+p]_q}{[n]_q}(1+r)}{[n+p+1]_q} r^{m-2}\right) + \frac{2m\frac{[n+p]_q}{[n]_q}(1+r)}{[n+p+1]_q} r^{m-1}$$
$$= r^2\left|D_{n,p}(e_{m-2};q,z) - e_{m-2}(z)\right| + \frac{2(m-1)\frac{[n+p]_q}{[n]_q}(1+r)}{[n+p+1]_q} r^{m-1} + \frac{2m\frac{[n+p]_q}{[n]_q}(1+r)}{[n+p+1]_q} r^{m}$$
$$= r^2\left|D_{n,p}(e_{m-2};q,z) - e_{m-2}(z)\right| + \frac{2\frac{[n+p]_q}{[n]_q}(1+r)}{[n+p+1]_q} r^{m-1}(2m-1) \leq \cdots$$
$$\leq \frac{\frac{[n+p]_q}{[n]_q}(1+r)}{[n+p+1]_q} r^{m-1} m(m+1).$$

Therefore, we have

$$\begin{aligned}\left|D_{n,p}(f;q,z)-f(z)\right| &\leq \sum_{m=1}^{\infty}|C_m|\left|D_{n,p}(e_m;q,z)-e_m(z)\right|\\ &\leq \sum_{m=1}^{\infty}|C_m|\frac{\frac{[n+p]_q}{[n]_q}(1+r)}{[n+p+1]_q}r^{m-1}m(m+1)\\ &\leq \frac{\frac{[n+p]_q}{[n]_q}(1+r)}{[n+p+1]_q}\sum_{m=1}^{\infty}|C_m|r^{m-1}m(m+1)\\ &\leq \frac{[n+p]_q}{[n]_q}\frac{C_r(f)}{[n+p+1]_q},\end{aligned}$$

where $C_r(f)=(1+r)\sum_{m=1}^{\infty}|C_m|r^{m-1}m(m+1)$.
This completes the proof of the theorem.

**Remark 3.1.** Let $0<q<1$ be fixed. Since, $\frac{1}{[n]_q}\to 1-q$ as $n\to\infty$, by applying limit $n\to\infty$, in the Theorem 3.1, $D_{n,p}(f;q,z)$ does not converge to $f(z)$ But this can be achieved by taking a sequence $q=q_n$ satisfying $0<q_n<1$ with $q_n\to 1$ and $q_n^n\to 0$ as $n\to\infty$. In this case $\frac{1}{[n]_q}\to 0$ as $n\to\infty$. Therefore from Theorem 3.1, we have $D_{n,p}(f;q,z)\to f(z)$ as $n\to\infty$ uniformly for $|z|\leq r$, when $\frac{[n+p]_q}{[n]_q}\leq r\leq R$.

Our next main result is the following Voronovskaja type theorem with a quantitative estimate.

**Theorem 3.2.** Let $\boldsymbol{R>1+p}$, $\boldsymbol{f: D_R\to C}$ be analytic in $\boldsymbol{D_R}$, i.e. we can write $\boldsymbol{f(z)=\sum_{k=0}^{\infty}c_k z^k}$, for all $\boldsymbol{z\in D_R}$. For any fixed $\boldsymbol{r\in [\frac{[n+p]_q}{[n]_q}, R)}$ and for all $\boldsymbol{n\in N}$, $\boldsymbol{|z|\leq r}$, we have

$$\boldsymbol{\left|D_{n,p}(f;q,z)-f(z)-\frac{z(1-z)f'(z)+z(1-(c+1)z)f'(z)}{[n]_q}\right|\leq \frac{1-q^{n+p}}{1-q^n}\frac{M_{r,p,k}(f)}{[n]_q^2}+\frac{T_{r,p,k}(f)}{[n]_q^2}+3(1-q)\sum_{k=1}^{\infty}|c_k|r^k k^4},$$

where $\boldsymbol{M_{r,p,k}(f)=\sum_{k=1}^{\infty}|c_k|r^k k F_{k,r,p}<\infty}$, $\boldsymbol{T_{r,p,k}(f)=\sum_{k=1}^{\infty}|c_k|r^k k D_{k,r,p}<\infty}$ and $\boldsymbol{F_{k,r,p}=4(1+c)k(k-1)^2(1+r)}$, $\boldsymbol{D_{k,r,p}=(1+p)(1+c)(2(k-1)^3+8k^2(k+1)+8k(k+1)^2)}$.

**Proof.** Since $\boldsymbol{f}$ is an analytic function, we can write $\boldsymbol{D_{n,p}(f;q,z)=\sum_{k=0}^{\infty}C_k D_{n,p}(e_k;q,z)}$. Also,

$$\frac{z(1-z)f'(z)+z(1-(c+1)z)f'(z)}{[n]_q} = \frac{z(1-z)}{[n]_q}\sum_{k=2}^{\infty} c_k k(k-1)z^{k-2} + \frac{z(1-(c+1)z)}{[n]_q}\sum_{k=1}^{\infty} c_k k z^{k-1} = \frac{1}{[n]_q}\sum_{k=1}^{\infty} c_k (k^2-(k+c)z)z^{k-1}$$

For all $z \in D_R$ and $n \in N$, we have

$$\left|D_{n,p}(f;q,z) - f(z) - \frac{z(1-z)f'(z)+z(1-(c+1)z)f'(z)}{[n]_q}\right| \le \sum_{k=1}^{\infty} c_k \left|D_{n,p}(e_k;q,z) - e_k(z) - \frac{(k^2-(k+c)z)}{[n]_q} z^{k-1}\right|.$$

If we consider $E_{k,n,p}(q,z) = D_{n,p}(e_k;q,z) - e_k(z) - \frac{(k^2-(k+c)z)}{[n]_q} z^{k-1}$, it is clear that $E_{k,n,p}(q,z)$ is a polynomial in $z$ of degree $\le k$. Using Lemma 2.4, we have

$$E_{k,n,p}(q,z) = \frac{q^k z(\frac{[n+p]_q}{[n]_q} - z)}{[n+p+k+1]_q} E'_{k-1,n,p}(q,z) + \frac{([k]_q \frac{[n+p]_q}{[n]_q} + q^k z[n+p]_q)}{[n+p+k+1]_q} E_{k-1,n,p}(q,z) + X_{k,n,p}(q,z),$$

where $X_{k,n,p}(q,z) = \frac{z^{k-2}}{[n]_q[n+p+k+1]_q}\{\frac{[n+p]_q}{[n]_q}(q^k\left(k-1\right)^2[k-2]_q + \left(k-1\right)^2[k]_q\right) + z\left(q^k[k-1]_q[n+p]_q - q^k(k-1)(k-1+c)[k-2]_q\frac{[n+p]_q}{[n]_q} - q^k(k-1)^2[k-2]_q + q^k(k-1)^2[n+p]_q + [k]_q[n+p]_q - q^k(k-1)(k-1+c)\frac{[n+p]_q}{[n]_q} - k^2[n+p+k+1]_q\right) + z^2\left(-q^k[n]_q[k-1]_q + q^k(k-1)(k-1+c)[k-2]_q + q^k[n]_q[n+p]_q + q^k(k-1)(k-1+c) - q^k[n+p]_q(k-1)(k-1+c) - [n]_q[n+p+k+1]_q + k(k+c)[n+p+k+1]_q\right)\Big\} = \frac{z^{k-2}}{[n]_q[n+p+k+1]_q}\left(A_{k,n}(q) + zB_{k,n}(q) + z^2C_{k,n}(q)\right).$

First we estimate $A_{k,n}(q)$.

$$A_{k,n}(q) = \frac{[n+p]_q}{[n]_q}(q^k(k-1)^2[k-2]_q + (k-1)^2[k]_q)$$

$$= \frac{1-q^{n+p}}{1-q^n}(q^k(k-1)^2[k-2]_q + (k-1)^2[k]_q).$$

It is clear from the above equation $|A_{k,n}(q)| \leq 2(1+p)(k-1)^3$. Next, we estimate $B_{k,n}(q)$. Using $[n+p]_q = [n]_q + q^n[p]_q$ and $[n+p+k+1]_q = [n]_q + q^n[p+k+1]_q$, we have

$$B_{k,n}(q) = [n]_q(q^k[k-1]_q + [k]_q + q^k(k-1)^2 - k^2) + [p]_q\left(q^{n+k}(k-1)^2 + q^n[k]_q + q^{n+k}[k-1]_q\right) - q^n k^2[p+k+1]_q + \frac{[n+p]_q}{[n]_q}(-q^k(k-1)^2[k-2]_q - q^k(k-1)(k-1+c)). \quad (7)$$

It is clear that

$$\left|\frac{[n+p]_q}{[n]_q}(-q^k(k-1)(k-1+c)[k-2]_q - q^k(k-1)(k-1+c))\right|$$

$$\leq \left|\frac{1-q^{n+p}}{1-q^n}\right| 2(1+c)\,(k-1)^3$$

$$\leq 2(1+c)(1+p)(k-1)^3.$$

Now, we have

$$[n]_q(q^k[k-1]_q + [k]_q + q^k(k-1)^2 - k^2) = (1-q^n)\{-q^k \textstyle\sum_{j=0}^{k-2}[j]_q - \sum_{j=0}^{k-1}[j]_q - k(k-1)[k]_q\},$$

which implies that

$$|[n]_q(q^k[k-1]_q + [k]_q + q^k(k-1)^2 - k^2)| \leq \frac{(k-1)(k-2)}{2} + \frac{(k-1)}{2} + k^2(k-1).$$

Hence, in view of (7), we have

$$|\mathbf{B}_{\mathbf{k,n}}(\mathbf{q})| \leq (\mathbf{1}+\mathbf{p})(\mathbf{1}+\mathbf{c})\{\mathbf{2}(\mathbf{k}-\mathbf{1})^{3}+\mathbf{k}(\mathbf{k}-\mathbf{1})^{2}+\mathbf{k}^{2}(\mathbf{k}-\mathbf{1})+(\mathbf{k}-\mathbf{2})(\mathbf{k}-\mathbf{1})^{2}+\mathbf{k}^{2}(\mathbf{k}-\mathbf{1})+\mathbf{k}^{2}(\mathbf{k}+\mathbf{1})+(\mathbf{k}-\mathbf{2})(\mathbf{k}-\mathbf{1})^{2}\} \leq \mathbf{8}(\mathbf{1}+\mathbf{p})(\mathbf{1}+\mathbf{c})\mathbf{k}^{2}(\mathbf{k}+\mathbf{1}).$$

Now, we estimate $\boldsymbol{C_{k,n}(q)}$. Using $[\boldsymbol{n}+\boldsymbol{p}]_{\boldsymbol{q}}=[\boldsymbol{n}]_{\boldsymbol{q}}+\boldsymbol{q}^{\boldsymbol{n}}[\boldsymbol{p}]_{\boldsymbol{q}}$ and $[\boldsymbol{n}+\boldsymbol{p}+\boldsymbol{k}+\mathbf{1}]_{\boldsymbol{q}}=[\boldsymbol{n}]_{\boldsymbol{q}}+\boldsymbol{q}^{\boldsymbol{n}}[\boldsymbol{p}+\boldsymbol{k}+\mathbf{1}]_{\boldsymbol{q}}$, we obtain

$$\boldsymbol{C_{k,n}(q)} \leq [\boldsymbol{n}]_{\boldsymbol{q}}^{2}(\boldsymbol{q}^{\boldsymbol{k}}-\mathbf{1})+[\boldsymbol{n}]_{\boldsymbol{q}}(-\boldsymbol{q}^{\boldsymbol{k}}[\boldsymbol{k}-\mathbf{1}]_{\boldsymbol{q}}+\boldsymbol{q}^{\boldsymbol{k}+\boldsymbol{n}}[\boldsymbol{p}]_{\boldsymbol{q}}-\boldsymbol{q}^{\boldsymbol{k}}(\boldsymbol{k}-\mathbf{1})(\boldsymbol{k}-\mathbf{1}+\boldsymbol{c})-\boldsymbol{q}^{\boldsymbol{n}}[\boldsymbol{p}+\boldsymbol{k}+\mathbf{1}]_{\boldsymbol{q}}+\boldsymbol{k}(\boldsymbol{k}+\boldsymbol{c}))+\boldsymbol{q}^{\boldsymbol{k}}[\boldsymbol{k}-\mathbf{2}]_{\boldsymbol{q}}(\boldsymbol{k}-\mathbf{1})(\boldsymbol{k}-\mathbf{1}+\boldsymbol{c})+\boldsymbol{q}^{\boldsymbol{k}}(\boldsymbol{k}-\mathbf{1})(\boldsymbol{k}-\mathbf{1}+\boldsymbol{c})-\boldsymbol{q}^{\boldsymbol{k}+\boldsymbol{n}}[\boldsymbol{p}]_{\boldsymbol{q}}(\boldsymbol{k}-\mathbf{1})(\boldsymbol{k}-\mathbf{1}+\boldsymbol{c})+\boldsymbol{q}^{\boldsymbol{n}}[\boldsymbol{p}+\boldsymbol{k}+\mathbf{1}]_{\boldsymbol{q}}\boldsymbol{k}(\boldsymbol{k}+\boldsymbol{c}).$$

Using the identites

$$[\boldsymbol{k}-\mathbf{1}]_{\boldsymbol{q}}=\sum_{\boldsymbol{j}=\mathbf{0}}^{\boldsymbol{k}-\mathbf{2}}[\boldsymbol{j}]_{\boldsymbol{q}}(\boldsymbol{q}-\mathbf{1})+(\boldsymbol{k}-\mathbf{1}), \quad [\boldsymbol{k}+\mathbf{1}]_{\boldsymbol{q}}=\sum_{\boldsymbol{j}=\mathbf{0}}^{\boldsymbol{k}}[\boldsymbol{j}]_{\boldsymbol{q}}(\boldsymbol{q}-\mathbf{1})+(\boldsymbol{k}+\mathbf{1}),$$

we obatin

$$[\boldsymbol{n}]_{\boldsymbol{q}}^{2}(\boldsymbol{q}^{\boldsymbol{k}}-\mathbf{1})+[\boldsymbol{n}]_{\boldsymbol{q}}\left(-\boldsymbol{q}^{\boldsymbol{k}}[\boldsymbol{k}-\mathbf{1}]_{\boldsymbol{q}}+\boldsymbol{q}^{\boldsymbol{k}+\boldsymbol{n}}[\boldsymbol{p}]_{\boldsymbol{q}}-\boldsymbol{q}^{\boldsymbol{k}}(\boldsymbol{k}-\mathbf{1})(\boldsymbol{k}-\mathbf{1}+\boldsymbol{c})-\boldsymbol{q}^{\boldsymbol{n}}[\boldsymbol{p}+\boldsymbol{k}+\mathbf{1}]_{\boldsymbol{q}}+\boldsymbol{k}(\boldsymbol{k}+\boldsymbol{c})\right)=[\boldsymbol{n}]_{\boldsymbol{q}}(\mathbf{1}-\boldsymbol{q}^{\boldsymbol{n}})\left(\boldsymbol{k}-[\boldsymbol{k}]_{\boldsymbol{q}}\right)+(\mathbf{1}-\boldsymbol{q}^{\boldsymbol{n}})\left\{\boldsymbol{q}^{\boldsymbol{k}}\sum_{\boldsymbol{j}=\mathbf{0}}^{\boldsymbol{k}-\mathbf{2}}[\boldsymbol{j}]_{\boldsymbol{q}}+\boldsymbol{q}^{\boldsymbol{k}}\sum_{\boldsymbol{j}=\mathbf{0}}^{\boldsymbol{k}}[\boldsymbol{j}]_{\boldsymbol{q}}+\boldsymbol{q}^{\boldsymbol{k}+\boldsymbol{n}}[\boldsymbol{p}]_{\boldsymbol{q}}+\boldsymbol{k}^{2}[\boldsymbol{k}]_{\boldsymbol{q}}-[\boldsymbol{k}]_{\boldsymbol{q}}\boldsymbol{k}(\mathbf{1}-\boldsymbol{c})\right\}+[\boldsymbol{n}]_{\boldsymbol{q}}(\boldsymbol{q}^{\boldsymbol{k}}\boldsymbol{c}-\boldsymbol{q}^{\boldsymbol{n}}).$$

Thus, we get

$$|\mathbf{C}_{\mathbf{k,n}}(\mathbf{q})| \leq \mathbf{10}(\mathbf{1}+\mathbf{p})(\mathbf{1}+\mathbf{c})\mathbf{k}(\mathbf{k}+\mathbf{1})^{2}+\mathbf{3}(\mathbf{1}+\mathbf{c})\mathbf{k}^{3}[\mathbf{n}]_{\mathbf{q}}(\mathbf{1}-\mathbf{q}^{\mathbf{n}}).$$

Therefore, we have

$$|\boldsymbol{X_{k,n,p}(q,z)}| \leq \frac{\boldsymbol{r}^{\boldsymbol{k}-\mathbf{2}}(\mathbf{1}+\boldsymbol{p})(\mathbf{1}+\boldsymbol{c})}{[\boldsymbol{n}]_{\boldsymbol{q}}^{2}}(\mathbf{2}(\boldsymbol{k}-\mathbf{1})^{3}+\mathbf{8}\boldsymbol{r}\boldsymbol{k}^{2}(\boldsymbol{k}+\mathbf{1})+\mathbf{10}\boldsymbol{r}^{2}\boldsymbol{k}(\boldsymbol{k}+\mathbf{1})^{2})+\mathbf{3}(\mathbf{1}+\boldsymbol{c})\boldsymbol{r}^{\boldsymbol{k}}\boldsymbol{k}^{3}(\mathbf{1}-\boldsymbol{q}).$$

From Theorem 3.1, we obtain

$$|\boldsymbol{D_{n,p}(e_k;q,z)}-\boldsymbol{e_k(z)}| \leq \frac{\mathbf{1}-\mathbf{q}^{\mathbf{n}+\mathbf{p}}}{\mathbf{1}-\mathbf{q}^{\mathbf{n}}}\frac{(\mathbf{1}+\mathbf{r})}{[\mathbf{n}]_{\mathbf{q}}}\mathbf{k}(\mathbf{k}+\mathbf{1})\mathbf{r}^{\mathbf{k}-\mathbf{1}}.$$

For all $\boldsymbol{k, n} \in \boldsymbol{N}$ and $|\boldsymbol{z}| \leq \boldsymbol{r}$ we get

$$|E_{k,n,p}(q,z)| \leq \frac{r\left(\frac{[n+p]_q}{[n]_q}+r\right)}{[n]_q} |E'_{k-1,n,p}(q,z)| + \frac{r[n+p]_q}{[n]_q} |E_{k-1,n,p}(q,z)| + |X_{k,n,p}(q,z)| \leq \frac{1-q^{n+p}}{1-q^n} \frac{r(1+r)}{[n]_q} |E'_{k-1,n,p}(q,z)| + r\frac{1-q^{n+p}}{1-q^n} |E_{k-1,n,p}(q,z)| + |X_{k,n,p}(q,z)|.$$

Now, we shall find an estimate of $|\boldsymbol{E'}_{k-1,n,p}(\boldsymbol{q},\boldsymbol{z})|$, for $\boldsymbol{k} \geq \boldsymbol{2}$

$$\begin{aligned} |E'_{k-1,n,p}(q,z)| &\leq \frac{k-1}{r} \left\|E_{k-1,n,p}\right\|_r \\ &\leq \frac{k-1}{r} ||D_{n,p}(e_{k-1}; q,z) - e_{k-1}(z)||_r \\ &+ \left|\left| \frac{((k-1)^2 - (k-1)(k-1+c)e_1)e_{k-2}}{[n]_q} \right|\right|_r \\ &\leq \frac{2(1+c)k(k-1)^2(1+r)}{[n]_q} r^{k-2} \end{aligned}$$

Thus, we have

$$|E_{k,n,p}(q,z)| \leq \frac{1-q^{n+p}}{1-q^n} \frac{4(1+c)k(k-1)^2(1+r)}{[n]_q^2} r^k + r\frac{1-q^{n+p}}{1-q^n} |E_{k-1,n,p}(q,z)| + |X_{k,n,p}(q,z)|,$$

where

$$|X_{k,n,p}(q,z)| \leq \frac{r^{k-2}(1+p)(1+c)}{[n]_q^2} (2(k-1)^3 + 8rk^2(k+1) + 10r^2k(k+1)^2) + 3(1+c)r^k k^3(1-q) \leq \frac{r^k}{[n]_q^2} D_{k,p,c} + 3(1+c)r^k k^3 (1-q),$$

where $D_{k,p,c} = (1+p)(1+c)\,(2(k-1)^3 + 8k^2(k+1) + 10k(k+1)^2)$.

Thus, for all $|z| \leq r, k \geq 1$ and $n \in N$, we obtain

$$|E_{k,n,p}(q,z)| \leq \frac{1-q^{n+p}}{1-q^n}\frac{4(1+c)k(k-1)^2(1+r)}{[n]_q^2}r^k + r\frac{1-q^{n+p}}{1-q^n}|E_{k-1,n,p}(q,z)| + \frac{r^k}{[n]_q^2}D_{k,p,c} + 3(1+c)r^k k^3(1-q) \leq \frac{1-q^{n+p}}{1-q^n}\frac{r^k}{[n]_q^2}F_{k,r,p} + r\frac{1-q^{n+p}}{1-q^n}|E_{k-1,n,p}(q,z)| + \frac{r^k}{[n]_q^2}D_{k,p,c} + 3(1+c)r^k k^3(1-q),$$

where $F_{k,r,p}$ is a polynomial of degree *3* in $k$ defined as

$F_{k,r,p} = 4(1+c)k(k-1)^2(1+r)$. But $E_{0,n,p}(q,z) = 0$, for any $z \in C$ and therefore writing the last inequality for $k = 1, 2,$ ...we easily obtain step by step following

$$|E_{k,n,p}(q,z)| \leq \frac{1-q^{n+p}}{1-q^n}\frac{r^k}{[n]_q^2}\sum_{j=1}^{k} F_{j,r,p} + \frac{r^k}{[n]_q^2}\sum_{j=1}^{k} D_{j,p,c} + 3(1+c)r^k(1-q)\sum_{j=1}^{k} j^3 \leq \frac{1-q^{n+p}}{1-q^n}\frac{r^k}{[n]_q^2}kF_{k,r,p} + \frac{r^k}{[n]_q^2}kD_{k,p,c} + 3(1+c)r^k(1-q)k^4.$$

Hence, we conclude that

$$\left|D_{n,p}(f;q,z) - f(z) - \frac{z(1-z)f'(z)+z(1-(c+1)z)f'(z)}{[n]_q}\right| \leq \sum_{k=1}^{\infty}|c_k||E_{k,n,p}(q,z)| \leq \frac{1-q^{n+p}}{1-q^n}\frac{1}{[n]_q^2}\sum_{k=1}^{\infty}|c_k|r^k kF_{k,r,p} + \frac{1}{[n]_q^2}\sum_{k=1}^{\infty}|c_k|r^k kD_{k,p,c} + 3(1+c)(1-q)\sum_{k=1}^{\infty}|c_k|r^k k^4.$$

As $\mathbf{f}^{(4)}(\mathbf{z}) = \sum_{k=4}^{\infty} c_k k(k-1)(k-2)(k-3)z^{k-4}$ and the series $\sum_{k=0}^{\infty} c_k z^k$ is absolutely convergent in $|z| \leq r$, it easily follows that $\sum_{k=4}^{\infty} c_k k(k-1)(k-2)(k-3)z^{k-4}$ is absolutely convergent in $|z| \leq r$, which implies that $\sum_{k=1}^{\infty}|c_k|r^k kF_{k,r,p} < \infty$ and $\sum_{k=1}^{\infty}|c_k|r^k kD_{k,p,c} < \infty$. Thus, the proof is completed**.**

**Remark 3.2.** Let $0 < q < 1$ be fixed. Since, $\frac{1}{[n]_q} \to 1 - q$ as $n \to \infty$, by applying limit $n \to \infty$, in the Theorem 3.2, we don't get the convergence. But this can be achieved by choosing $1 - \frac{1}{n^2} \leq q_n < 1$ with with $q_n \to 1$ as $n \to \infty$. In this case $\frac{1}{[n]_{q_n}} \to 0$ as $\to \infty$ , $\frac{1-q^{n+p}}{1-q^n} \to 1$ as $n \to \infty$ and $1 - q_n \leq \frac{1}{n^2} \leq \frac{1}{[n]_{q_n}^2}$ Therefore, from Theorem 3.2, we have

$|D_{n,p}(f;q,z)-f(z)-\frac{z(1-z)f'(z)+z(1-(c+1)z)f'(z)}{[n]_q}| \le \frac{M_{r,p,k}(f)}{[n]_{q_n}^2}+\frac{T_{r,p,k}(f)}{[n]_{q_n}^2}+$
$\frac{3}{[n]_{q_n}^2}\sum_{k=1}^{\infty}|c_{k|}r^k k^4$,

that is, the order of approximation is $O(\frac{1}{[n]_{q_n}^2})$.

Now we will find the exact order of approximation for complex q-modified Bernstein-Schurer operators. Throughout thesection, we assume that $q_n$ is a sequence such that $0<q_n<1$ with $q_n\to 1$ as $n\to\infty$ and $q_n^n\to a(a<1)$ as $n\to\infty$.

**Theorem 3.3.** Let $R>1+p$, $D_R=\{z\in C;|z|<R\}$ and $f(z)=\sum_{m=0}^{\infty}c_m z^m$, for all $z\in D_R$. If $f$ is a non constant polynomial, then for $r\in[\frac{[n+p]_q}{[n]_q},R)$

$$||D_{n,p}(f;q_n,.)-f||_r \ge \frac{C'_{r,n,p}(f)}{[n]_q},$$

where $C'_{r,p}(f)>0$ depends on $f,r,p$ and on the sequence $\{q_n\}_{n\in N}$ but it is independent on of $n$.

**Proof.** For all $z\in D_R$ and $n\in N$, we get

$$||D_{n,p}(f;q_n,.)-f||_r=\frac{1}{[n]_q}\{z(1-z)f''(z)+(1-(c+1)z)f'(z)+$$
$$\frac{1}{[n]_q}([n]_{q_n}^2(D_{n,p}(f;q,z)-f(z)-\frac{z(1-z)f'(z)+z(1-(c+1)z)f'(z)}{[n]_q}))\}.$$

Now, using the identity $||F+G||_r\ge|||F||_r-||G||_r|\ge||F||_r-||G||_r$, we obtain

$$\begin{aligned}&||D_{n,p}(f;q_n,.)-f||_r\\ &\quad\ge\frac{1}{[n]_q}\{e_1(1-e_1)f''+(1-(c+1)e_1)f'\\ &\quad-\frac{1}{[n]_q}([n]_{q_n}^2||D_{n,p}(f;q_n,.)-f\\ &\quad-\frac{e_1(1-e_1)f''+(1-(c+1)e_1)f''}{[n]_q}||)\}.\end{aligned}$$

Since $f$ is a non-constant polynomial in $D_R$ ,we get

$$||e_1(1-e_1)f''+(1-(c+1)e_1)f'||_r>0.$$

Indeed, supposing contrary it follows that

$$z(1-z)f''(z)+(1-(c+1)z)f'(z)=0,$$

writing the expansion of $f'(z)$ and $f''(z)$ in the last equality, we can see that $a_m=0, m=1,2,\ldots$ Thus $f$ is a constant, which is a contradiction to the hypothesis.

Now, from Remark 3.2, we have
$[n]_{q_n}^2 |D_{n,p}(f;q_n,.)-f-\frac{e_1(1-e_1)f''+(1-(c+1)e_1)f'}{[n]_q}| \to 0$ *as* $n\to\infty$.
Consequently, there exists $n_1$ (depending only on $f$and r) such that, for all $n\geq n_1$, we have
$||e_1(1-e_1)f''+(1-(c+1)e_1)f'||_r-\frac{1}{[n]_q}([n]_{q_n}^2||D_{n,p}(f;q_n,.)-f-\frac{e_1(1-e_1)f''+(1-(c+1)e_1)f'}{[n]_q}||)\geq\frac{1}{2}||e_1(1-e_1)f''+(1-(c+1)e_1)f'||_r,$
which implies that
$||D_{n,p}(f;q_n,.)-f||_r\geq\frac{1}{2[n]_{q_n}}||e_1(1-e_1)f''+(1-(c+1)e_1)f'||_r,$ for all $n\geq n_1$. For $1\leq n\leq n_1$, we have
$||D_{n,p}(f;q_n,.)-f||_r\geq\frac{C_{r,n,p}(f)}{2[n]_{q_n}},$ *with* $C_{r,n,p}(f)=[n]_{q_n}||D_{n,p}(f;q_n,.)-f||_r>0$.
Then, finally we get

$$||D_{n,p}(f;q_n,.)-f||_r\geq\frac{\mathbf{C}'_{\mathbf{r,n,p}}(\mathbf{f})}{[n]_q},$$

where $\mathbf{C}'_{\mathbf{r,n,p}}(\mathbf{f})=\min\{C_{r,1,p},C_{r,2,p}\ldots C_{r,n-1,p},\frac{1}{2}||e_1(1-e_1)f''+(1-(c+1)e_1)f'||_r\}$. Hence, the proof is completed.

## 4. Conclusion

In this paper, an upper bound, the exact order of approximation and a Voronovskaja-type theorem with a quantitative estimate are obtained for the complex q-Bernstein-Durrmeyer type operators attached to analytic functions on compact disks. Our results show that extension of complex q-Bernstein-Durrmeyer type operators from real intervals to compact disks in the complex plane extends approximation properties.

## R E F E R E N C E S

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