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d- STATISTICAL CONVERGENCE OF ORDER α AND d-STATISTICAL BOUNDEDNESS OF ORDER α IN METRIC SPACES

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In the present paper, we introduce and study d- statistical convergence of order α d-statistical boundedness of order α and d- strong p -Cesàro summability of order α for sequences in a metric space. Furthermore, we investigate the relations between the sets of sequences which are d-statistically convergent of order α, between the sets of sequences which are d- statistically bounded of order α and between the sets of sequences which are d- strongly p -Cesàro summable of order α for various values of α's in (0,1].

Keywords: α- density; statistical convergence; statistical convergence of order α; statistical boundedness of order α; strong $p-$ Cesàro summability of order α.

1. Introduction

The main topic of this paper is to study the statistical convergence of order α, the statistical boundedness of order α $(\alpha \in (0,1])$ and strong p- Cesàro summability of order α $(\alpha > 0)$ for sequences in metric spaces. We will start by saying a few words about the history of these concepts which are also related to each other.

The thinking of statistical convergence was first given by Zygmund [1] in 1935. Statistical convergence was introduced for the first time by Steinhaus [2] and Fast [3] and then by Schoenberg [4] independently. Some authors studied also the statistical convergence of a sequence along density of the sets of natural numbers that we could mention R. C. Buck [5] for instance. In the last decades and under different names the subject was discussed in many different theories such as in the theory of Fourier analysis, number theory, ergodic theory, measure theory, trigonometric series and Banach spaces. It was further investigated from the sequence spaces and summability theory point of view and via summability theory by Fridy [6], Connor [7], Mursaleen [8], Salat [9], Bhardwaj [10] and many others.

From beginning to the last years, the statistical convergence has been defined and investigated for the sequences of real or complex numbers. In this

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paper we study these concepts in metric spaces with order of a real number α between 0 and 1.

The order of statistical convergence of a sequence of positive linear operators was introduced by Gadjiev and Orhan [11]. The statistical convergence of order α $(0 < \alpha \le 1)$ and strong p-Cesàro summability of order α were introduced and studied by Çolak [12] for number sequences, using the notion α-density of a subset of the set $\mathbb{N}$ of positive integers.

For detailed investigations on the concept concerning statistical convergence of order α see ([13], [14], [15]).

The sequence space $w = \left\{ x = (x_k) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - l| = 0, \text{ for some number } l \right\}$ of strongly Cesàro summable number sequences were introduced and studied by Maddox [16].

Recently, Çolak [12] introduced the strong p-Cesàro summability of order α where $0 < \alpha \le 1$ and p is a positive real number, which is a generalization of the strong Cesàro summability. Furthermore the relations between the sets of sequences which are statistical convergent of order α and the sets of sequences which are strongly p- Cesàro summable of order α were given for various values of α in [12].

We now recall some definitions which will be needed in the sequel of this paper.

A number sequence $x = (x_k)$ is said to be statistically convergent to a number l if for each $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - l| \ge \varepsilon\}$ has the natural density zero, where the natural density of a subset $G \subset \mathbb{N}$ is defined by $\delta(G) = \lim_{n\to\infty} \frac{1}{n} |\{k \le n : k \in G\}|$ (see [17]) and $|\{k \le n : k \in G\}|$ denotes the number of elements of G not exceeding *n*.

Definition 1.1 ([12]) Let $\alpha \in (0,1]$ be any real number. The α***-density*** of a subset $H \subset \mathbb{N}$ is defined by $\delta_\alpha(H) = \lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \le n : k \in H\}|$, where the limit exists (finite or infinite).

Obviously $\delta_\alpha(\mathbb{N}) = 1$ $(\alpha = 1)$, $= \infty$ $(\alpha < 1)$ and we have $\delta_\alpha(H) = 0$ for every $\alpha \in (0,1]$ provided that $H \subset \mathbb{N}$ is a finite subset. Although $\delta_\alpha(H^c) = 1 - \delta_\alpha(H)$ for $\alpha = 1$, the equality $\delta_\alpha(H^c) = 1 - \delta_\alpha(H)$ is not true for $0 < \alpha < 1$ in general. Note that for every subset $H \subseteq \mathbb{N}$, $\delta_\alpha(H) = 0$ if $\alpha > 1$.

Also the α– density $\delta_\alpha(H)$ reduces to the natural density $\delta(H)$ of a subset $H \subset \mathbb{N}$ in case $\alpha = 1$.

Lemma 1.2. ([12]) Let $E \subseteq \mathbb{N}$. Then $\delta_\beta(E) \le \delta_\alpha(E)$ if $0 < \alpha \le \beta \le 1$.

Now we give different versions of the well-known definitions of convergence and boundedness of a sequence in a metric space.

Definition 1.3. A sequence (x_k) in a metric space (X, d) is said to be convergent if there exists a point $x_o \in X$ such that for every $\varepsilon > 0$ there exists a real number N_ε such that $|\{k \in \mathbb{N} : x_k \notin B_\varepsilon(x_o)\}| \le N_\varepsilon$, where $B_\varepsilon(x_o) = \{x \in X : d(x, x_o) < \varepsilon\}$ is the open ball of radius ε and center x_o.

Definition 1.4. The sequence (x_k) in a metric space (X, d) is bounded if there exist a point $x \in X$ and a positive real number M such that $|\{k \in \mathbb{N} : x_k \notin B_M(x)\}| = 0$.

2. *d*-statistical convergence of order α and *d*- statistical boundedness of order α in a metric space

The statistical convergence of order α $(0 < \alpha \le 1)$ was studied by Çolak [12] for number sequences, using the $\alpha-$ density of subsets of $\mathbb{N}$. Statistical convergence in metric spaces was studied by Küçükarslan et al. [18]. In this section we study statistical convergence of order α and statistical boundedness of order α in metric spaces.

Definition 2.1 Let (X, d) be a metric space, $x = (x_k)$ be any sequence in this space and let $0 < \alpha \le 1$ be given. The sequence is said to be $d-$ *statistically convergent of order* α if there is a point $x_o \in X$ such that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \le n : x_k \notin B_\varepsilon(x_o)\}| = 0$$

that is $\delta_\alpha(\{k \in \mathbb{N} : x_k \notin B_\varepsilon(x_o)\}) = 0$ for every $\varepsilon > 0$. If the sequence (x_k) is statistically convergent of order α, to $x_o \in X$ we write $S_d^\alpha - \lim x_k = x_o$.

The statistical convergence of order α reduces to the statistical convergence for $\alpha = 1$ in a metric spaces [12]. The set of all sequences which are $d-$ statistically convergent of order α in the metric space (X, d) will be denoted by S_d^α and we will write S_d for the set of all $d-$ statistically convergent sequences in case $\alpha = 1$.

Lemma 2.2. Let $\alpha \in (0,1]$ be given. If a sequence (x_k) is $d-$ statistically convergent of order $\alpha,$ then its $S_d^\alpha -$ limit is unique.
The proof is easy so we omit it.

Remark 2.3. The $d-$ statistical convergence of order α is well defined only for $0<\alpha \leq 1$. In order to show this let (X,d) be a metric space and let $x=(x_k)$ be a sequence such that x_k = *a (k=2n), = b (k≠2n),* where $a,b \in X$ are two fixed points and $a \neq b$. Then it is easy to see that the sequence $x=(x_k)$ is $d-$ statistically convergent of order α , both to a and to b i.e. $S_d^\alpha - \lim x_k = a$ and $S_d^\alpha - \lim x_k = b$ for any $\alpha > 0$. But this contradicts Lemma 2.2.

Remark 2.4. One may see that every convergent sequence is $d-$ statistically convergent of order α , that is $c_d \subset S_d^\alpha$ for each $0<\alpha \leq 1$ in a metric space (X,d) . Easily it can be seen that the converse of this fact is not true. For example, the sequence $x=(x_k)$ defined by

$$x_k = \begin{cases} a, & k=n^3 \\ b, & k \neq n^3 \end{cases} \quad n=1,2,3,\ldots$$

with $a,b \in X$ and $a \neq b$, is $d-$ statistically convergent of order α for $\alpha > \frac{1}{3}$ ($S_d^\alpha - \lim x_k = b$), but it is not convergent.

Theorem 2.5. Let (X,d) be a metric space and $0<\alpha \leq \beta \leq 1$. Then $S_d^\alpha \subseteq S_d^\beta$ and the inclusion is strict for some α and β if there is $k \in \mathbb{N}$ such that $\alpha < \frac{1}{k} < \beta$.

Proof. Let (X,d) be a metric space, $x=(x_k) \in S_d^\alpha$ and $0<\alpha \leq \beta \leq 1$. Then we may write

$$\frac{1}{n^\beta}\left|\{k \leq n \ : \ x_k \notin B_\varepsilon(x_o)\}\right| \leq \frac{1}{n^\alpha}\left|\{k \leq n \ : \ x_k \notin B_\varepsilon(x_o)\}\right|$$

for every $\varepsilon > 0$ and this gives that $S_d^\alpha \subseteq S_d^\beta$. In order to show that the inclusion is strict (see also Theorem 2 in [13]) we may consider the following example:

Example 2.6. Take $X = l_\infty$ with the metric *d(a,b)*= sup $|a_k - b_k|$, where $a=(a_k)$**,** $b=(b_k) \in l_\infty$, set of bounded sequences. Consider the sequence (x^k) where $x^k = (x_i^k)_{i=1}^\infty \in l_\infty$ is defined by

$$x_i^k = \begin{cases} \frac{1}{k}, & \text{for each } i=1,2,3,\ldots \text{ if } k=n^2 \\ 0, & \text{for each } i=1,2,3,\ldots \text{ if } k \neq n^2 \end{cases} \quad n=1,2,3,\ldots .$$

Now we may write

$$\frac{1}{n^{\beta}}\left|\left\{k \le n \,:\, d\left(x^{k},\theta\right)=\sup_i\left|x_i^k-0\right|\ge\varepsilon\right\}\right| \le \frac{1}{n^{\beta}}\sqrt{n}.$$

where $\theta=(0,0,...)$. Then taking the limit as $n\to\infty$ we obtain $\left(x^k\right)\in S_d^{\beta}$ for $\frac{1}{2}<\beta\le 1$, but $\left(x^k\right)\notin S_d^{\alpha}$ for since

$$\frac{\sqrt{n}-1}{n^{\alpha}}\le\frac{1}{n^{\alpha}}\left|\left\{k \le n \,:\, d\left(x^{k},\theta\right)=\sup_i\left|x_i^k-0\right|\ge\varepsilon\right\}\right|.$$

If we take $\beta=1$ in Theorem 2.5 we have the following result.

Corollary 2.7. If a sequence in a metric space is d- statistically convergent of order $\alpha\in(0,1]$ to a point $x_o\in X$, then it is d-statistically convergent to x_o, that is $S_d^{\alpha}\subseteq S_d$ and the inclusion is strict if 0< α <1.

Theorem 2.8. Let d and d' be two metrics on X and $0<\alpha\le 1$ be given. If $d\succ d'$ that is $d(a,b)\ge d'(a,b)$ for every $a,b\in X$ then $S_d^{\alpha}\subseteq S_{d'}^{\alpha}$.
The proof is easy so we omit it.

Definition 2.9. Let (X,d) be a metric space and $0<\alpha\le 1$ be given. A sequence $x=(x_k)$ in the metric space (X,d) is called $d-$ statistically bounded of order α if there exist a point $x\in X$ and a real number $M>0$ such that

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}\left|\left\{k\le n \,:\, d(x_k,x)\ge M\right\}\right|=0.$$

The set of sequences which are d- statistically bounded of order α in metric space (X,d) will be denoted by BS_d^{α}.
In case $\alpha=1$, d - statistical boundedness of order α reduces to statistical boundedness. The set of statistically bounded sequences will be denoted by BS_d [19].
Theorem 2.10. Any bounded sequence in a metric space (X,d) is d - statistically bounded of order α for each $\alpha\in(0,1]$.

Proof Assume that $x=(x_k)$ is a bounded sequence in a metric space (X,d) and let $\alpha\in(0,1]$ be given. Since the sequence (x_k) is bounded then there exist a real number $M>0$ and a point $x\in X$ such that $d(x_k,x)<M$ for every $k\in\mathrm{N}$. Then

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}\left|\left\{k\le n \,:\, d(x_k,x)\ge M\right\}\right|=0$$

for any $\alpha \in (0,1]$, since $\{k \le n : d(x_k, x) \ge M\} = \phi$ for each $n \in \mathbb{N}$. Therefore the sequence (x_k) is $d-$ statistically bounded of order α for each $\alpha \in (0,1]$. This completes the proof.

Remark 2.11. The converse of Theorem 2.10 is not true. For this, let us consider the metric space $X = \mathbb{R}$ with the usual metric. The sequence (x_k) defined by

$$x_k = \begin{cases} k, & k = m^2 \\ (-1)^k, & k \ne m^2 \end{cases} \quad m=1,2,3,\ldots$$

is not bounded. But since the inequality

$$\frac{1}{n^\alpha}\left|\{k \le n : |x_k| \ge M\}\right| \le \frac{\sqrt{n}+1}{n^\alpha}$$

is satisfied for a sufficiently large $M > 0$ and the right side of the last inequality tends to 0 as $n \to \infty$ for $\alpha \in \left(\frac{1}{2},1\right]$, we obtain that the sequence (x_k) is $d-$ statistically bounded of order α.

Theorem 2.12. Let (X,d) be a metric space and let $0 < \alpha \le \beta \le 1$ be given.
If a sequence $x = (x_k)$ in *X* is *d*- statistically convergent of order α , then it is $d-$ statistically bounded of order β that is $S_d^\alpha \subseteq BS_d^\beta$.
If a sequence $x = (x_k)$ in X is $d-$ statistically bounded of order α then it is $d-$ statistically bounded of order β that is $BS_d^\alpha \subseteq BS_d^\beta$.
Proof (i) Let $0 < \alpha \le \beta \le 1$ be given. Assume that $x = (x_k)$ is $d-$ statistically convergent of order α to x_o. Then for every $\varepsilon > 0$ and a large $M > 0$ we may write

$$\{k \le n : x_k \notin B_M(x_o)\} \subset \{k \le n : x_k \notin B_\varepsilon(x_o)\}.$$

From this inclusion we obtain the inclusion $S_d^\alpha \subseteq BS_d^\beta$.
Let $(x_k) \in BS_d^\alpha$. For a sufficiently large number $M > 0$ we may write

$$\frac{1}{n^\beta}\left|\{k \le n : x_k \notin B_M(x_o)\}\right| \le \frac{1}{n^\alpha}\left|\{k \le n : x_k \notin B_M(x_o)\}\right|.$$

Since $x = (x_k) \in BS_d^\alpha$, the right hand side of the above inequality tends to 0 as $n \to \infty$ and thus the left hand side tends to 0 and therefore $BS_d^\alpha \subseteq BS_d^\beta$ and this completes the proof.

Corollary 2.13. In a metric space

(i) Every $d-$ statistically convergent sequence of order α is $d-$ statistically bounded of order , that is ,

(ii) Every $d-$ statistically convergent sequence of order α is $d-$ statistically bounded, that is $S_d^{\alpha} \subseteq BS_d$,

(iii) Every $d-$ statistically bounded sequence of order α is $d-$ statistically bounded, that is $BS_d^{\alpha} \subseteq BS_d$ for each $\alpha \in (0,1]$.

(iv) Every $d-$ statistically convergent sequence is $d-$ statistically bounded, that is $S_d \subseteq BS_d$.

Corollary 2.13 *(iv)* is Theorem 1 *(ii)* of Küçükarslan and Değer [19] which we obtain taking $\alpha = \beta = 1$ in Theorem 2.12 *(i)* .

3 - Strong *p* - Cesàro summability of order α in a metric space

In this section we study *d* - strong *p* - Cesàro summability of order α and give the relations between the sets of sequences which are $d-$ strongly *p*- Cesàro summable of order α for various values of α in $(0,+\infty)$ in metric spaces.

Definition 3.1 Let (X,d) be a metric space and let $\alpha > 0$, $p > 0$ be real numbers. A sequence $x = (x_k)$ in X is said to be $d-$*strongly* p - Cesàro summability of order α, if there is a point $x_o \in X$ such that

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} \sum_{k=1}^{n} [d(x_k, x_o)]^p = 0.$$

The set of all sequences which are $d-$ strongly $p-$ Cesàro summable of order α in the metric space (X,d) will be denoted by w_{pd}^{α} and we write w_{pd} instead of w_{pd}^{α} for $\alpha = 1$. $d-$ strong $p-$ Cesàro summability of order α reduces to $d-$ strong $p-$ Cesàro summability for $\alpha = 1$ (see [20]).

Theorem 3.2. Let (X,d) be a metric space, $0 < \alpha \le \beta$ and $p > 0$ be a real number. Then, $w_{pd}^{\alpha} \subseteq w_{pd}^{\beta}$ and the inclusion may be strict if $\alpha < \beta$.

The proof of inclusion is straightforward. We may consider the following example in order to show that the inclusion is strict.

Example 3.3. Take $X = \mathbb{R}$ with $d(x,y) = |x - y|$ and use the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 0, & k \ne m^2 \\ 1, & k = m^2 \end{cases} m=1,2,3,\dots .$$

We may write the inequality

$$\frac{1}{n^{\beta}} \sum_{k=1}^{n} |x_k - 0|^p \le \frac{\sqrt{n}}{n^{\beta}} = \frac{1}{n^{\beta-\frac{1}{2}}}.$$

Since $\frac{1}{n^{\beta-\frac{1}{2}}} \to 0$ as $n \to \infty$, then $w_{pd}^{\beta} - \lim x_k = 0$, i.e. $x \in w_{pd}^{\beta}$ for $\frac{1}{2} < \beta$, but since

$$\frac{\sqrt{n}-1}{n^{\alpha}} \leq \frac{1}{n^{\alpha}} \sum_{k=1}^{n} |x_k - 0|^p$$

and $\frac{\sqrt{n}-1}{n^{\alpha}} \to \infty$ as $n \to \infty$, then $x \notin w_{pd}^{\alpha}$ for $0 < \alpha < \frac{1}{2}$. This completes the proof.
If we take $\beta = 1$ in Theorem 3.2 we may easily obtain the following result.

Corollary 3.4. Let (X, d) be a metric space and $0 < p < \infty$. Then $w_{pd}^{\alpha} \subseteq w_{pd}$ for every $\alpha > 0$.

4. Some Inclusion Relations between S_d^{α} and w_{pd}^{α} in a metric space

In this section we give some relations between the sets of sequences which are $d-$ statistically convergent of order α and the sets of sequences which are $d-$ strongly $p-$ Cesàro summable of order α for various values of α in $(0,1]$ in metric spaces.

Theorem 4.1 Let (X, d) be a metric space, α and β be two fixed real numbers with $0 < \alpha \leq \beta \leq 1$ and $0 < p < \infty$. If a sequence in X is $d-$ strongly $p-$ Cesàro summable of order α to x_o , then it is $d-$ statistically convergent of order β to x_o .
Proof. Let (X, d) be a metric space. For any sequence $x = (x_k)$ in X and $\varepsilon > 0$ we may write

$$\frac{1}{n^{\alpha}} \sum_{k=1}^{n} [d(x_k, x_o)]^p \geq \frac{1}{n^{\alpha}} |\{k \leq n \ : \ x_k \notin B_{\varepsilon}(x_o)\}| \varepsilon^p \geq \frac{1}{n^{\beta}} |\{k \leq n \ : \ x_k \notin B_{\varepsilon}(x_o)\}| \varepsilon^p .$$

Hence taking the limit as $n \to \infty$ in this inequality, it follows that if $x = (x_k)$ is $d-$ strongly $p-$ Cesàro summable of order α, to x_o , then it is $d-$ statistically convergent of order β, to x_o and this completes the proof.
Taking the special cases $\beta = \alpha$ and $\beta = \alpha = 1$ in Theorem 4.1 we obtain (i) and (ii) of the following Corollary 4.2, respectively.

Corollary 4.2. Let (X, d) be a metric space and $0 < p < \infty$.
(i) If a sequence in metric space (X, d) is $d-$ strongly $p-$ Cesàro summable of order α to x_o , then it is $d-$ statistically convergent of order α to x_o for every $\alpha \in (0,1]$.

(ii) ([20]) If a sequence is $d-$ strongly $p-$ Cesàro summable to x_o , then it is $d-$ statistically convergent to x_o .

Remark 4.3. The converse of Corollary 4.2 (i) and (ii) does not hold in general, so the converse of Theorem 4.1 does not hold.

Example 4.4. Consider the metric space $X=\mathbb{R}$ with metric $d(x,y)=|x-y|$ and the sequence (x_k) defined as

$$x_k=\begin{cases}\left(1+\frac{1}{k}\right)^{k^2}, & k=n^3\\ 0, & k\neq n^3\end{cases}\quad n=1,2,\ldots .$$

For every $\varepsilon>0$, since

$$\frac{1}{n^\alpha}\left|\{k\le n\ :\ d(x_k,0)\ge\varepsilon\}\right|\le\frac{1}{n^\alpha}\sqrt[3]{n},$$

then $(x_k)\in S_d^\alpha$ for $\frac{1}{3}<\alpha\le 1$. However, in case $p=1$ we may write

$$\sum_{k=1}^{n}[d(x_k,0)]^p=\sum_{k=1}^{n}|x_k|=\sum_{\substack{k=1\\k=m^3}}^{n}\left(1+\frac{1}{k}\right)^{k^2}+\sum_{\substack{k=1\\k\neq m^3}}^{n}0=\sum_{\substack{k=1\\k=m^3}}^{n}\left(1+\frac{1}{k}\right)^{k^2}. \tag{4.1}$$

If we apply the well-known Bernoulli Inequality ($(1+a)^n\ge 1+na$ for all $n\in\mathbb{N}$, where $a>-1,\ a\in\mathbb{R}$) then continuing from the statement (4.1) we may write

$$\sum_{k=1}^{n}[d(x_k,0)]^p\ge\sum_{\substack{k=1\\k=m^3}}^{n}\left(1+k^2\frac{1}{k}\right)=\sum_{\substack{k=1\\k=m^3}}^{n}(1+k)\ge\sum_{\substack{k=1\\k=m^3}}^{n}k\ge 1+2^3+3^3+\ldots+\left(\left[\sqrt[3]{n}\right]\right)^3=\left[\frac{\left[\sqrt[3]{n}\right]\left(\left[\sqrt[3]{n}\right]+1\right)}{2}\right]^2$$

where $[r]$ is the integer part of the real number r. Hence we get

$$\frac{1}{n^\alpha}\sum_{k=1}^{n}[d(x_k,0)]^p=\frac{1}{n^\alpha}\sum_{k=1}^{n}|x_k|^p=\frac{1}{n^\alpha}\sum_{k=1}^{n}|x_k|\ge\frac{1}{n^\alpha}\frac{\left[\sqrt[3]{n}\right]^4+2\left[\sqrt[3]{n}\right]^3+\left[\sqrt[3]{n}\right]^2}{4}>\frac{1}{n^\alpha}\frac{\left[\sqrt[3]{n}\right]^4}{4}.$$

This gives that $(x_k)\notin w_{pd}^\alpha$ for $\alpha<\frac{4}{3}$ and hence for $0<\alpha\le 1$. Consequently, $(x_k)\in S^\alpha-w_{pd}^\alpha$ for $\frac{1}{3}<\alpha\le 1$.

Remark 4.5 We see that in a metric space, a bounded and statistically convergent sequence of order α needs not be strongly $p-$ Cesàro summable of order α , in general, for $0<\alpha<1$. To show this we give the following example.

Example 4.6. Take the metric space $X=\mathbb{R}$ with $d(x,y)=|x-y|$. The sequence $x=(x_k)$ defined by

$$x_k = \begin{cases} \frac{1}{\sqrt{k}}, & k \neq m^3 \\ 1, & k = m^3 \end{cases}$$

is an example for this case. It is clear that $x \in l_\infty$ and it can be shown that $x \in S^\alpha - w_{pd}^\alpha$ for $\frac{1}{3} < \alpha < \frac{1}{2}$ if $p = 1$ (see [12]).

Corollary 4.7. Let (X, d) be a metric space, $0 < \alpha \leq 1$ and let p be a positive real number. Then, $w_{pd}^\alpha \subset S_d$. and the inclusion may be strict for $0 < \alpha < 1$.
Proof follows from Corollary 4.2 and Corollary 2.7.

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