U.P.B. Sci. Bull., Series D, Vol. 71, Iss. 1, 2009 ISSN 1454-2358

ON THE OPTIMAL DESIGN OF HELICAL SPRINGS OF AN AUTOMOBILE SUSPENSION

Aurel P. STOICESCU[1]

In lucrare se prezintă metoda de proiectare optimală a arcurilor elicoidale din suspensia automobilelor dupa criteriul masei lor minime. Pentru aceasta, la o rigiditate dată a arcului, sunt luate în considerare solicitarea corespunzătoare forței maxime preluate de arc, solicitarea la oboseală, condția de stabilitate la flambaj, restricțiile privitoare la indicele şi diametrul exterior ale arcului. Exemplul numeric tratat în lucrare permite să se tragă concluzii cu caracter mai general.

The paper presents the optimal design method of the helical springs of the automobile suspensions according to the criterion of the minimum mass. For this purpose, at a given spring rate, the torsional stress corresponding to the maximum force applied to the spring, the fatigue stress, the buckling stability condition and the constraints relating to the spring index and to the outer coil diamete are considered. The work example allows also to draw more general conclusions.

Key words: automobile suspension, helical spring, optimal design, spring mass

1. Introduction

In most cases, the main spring element of the independent wheel suspension is the helical (coil) spring. This should ensure a given suspension stiffness that determines the natural frequency of the oscillations of the automobile sprung mass. It is a fundamental quantity that characterizes smoothness of the ride [1- 9]. At the same time, the coil spring should ensure large deflections in order to corresponde to the highest allowed displacements of the wheels in the relative movement to the body. The all above mentioned items are rendered evident on the elastic characteristic of the suspension. The establishing of the elastic characteristic parameters is made according to the literature indications (see especially [5] and [9]) that rely in part on theoretical considerations and to a great extent on numerous empirical data.

If we know the coil spring stiffness and the maximum deflection we can determine the coil spring dimensions by making use of the known methods for the stress and displacement calculation [10, 11,12]. As a rule, in textbooks and other bibliographical sources [1, 2, 13] mentions are made of the above given kind without dwelling upon the peculiarity of the given case. There are works however

[1] Prof., Motor Vehicle Dept., University Politehnica of Bucharest, ROMANIA

[3,5,9] that render these peculiarities evident and present concretely how to calculate the helical spring dimensions. But even in this case it is necessary to choose initialy two quantities in the following variants: the mean spring diameter D and the spring index $i=D/d$ (d-coil or wire diameter) [3]; the mean spring diameter D and roughly the coil diameter d [5]; the coil diameter and the spring index; the number of working coils and the gear ratio of the spring to the wheel [9]. The values of these quantities are chosen in accord with one of the following cases: a) taking into account the available space (in this case, one chooses the mean spring diameter}; b) one chooses the spring index that belongs to a given interval; c) one uses the specifications of the similar automobiles. But, in general, the literature does not show how to rationally choose the above mentioned quantities. For example, it is important to know if it is really justified to choose the maximum value of the mean spring diameter in the way that is suggested in [3] (it is assumed that there is a given known available space).

Still it is necessary to add the fact that because of the nonliniarities of the used expressions the spring calculation is iterativ. In general, the strategy of the iteration performing is not enough justified.

From all those facts it results that and an additional condition may be posed or that certain optimization criteria may be considered. One of those criteria could be the spring mass. The condition consists in the minimization of this mass. In general, the problem of mass minimization of a helical spring is briefly formulated in [13], where the complete solution of a torsional spring is presented. Also, in [12] some works are refered that deal with a similar problem in the case of the valve springs of the internal combustion engines.

The objective of this paper is to describe a method of the optimal design of an automobile suspension coil spring according to mass minimization criterion. Therefore, one uses the theory of the nonlinear programming with constraints. As a preliminary, a series of correction coefficients that are ordinarely given by diagrams are expressed analytically. At that same time one discusses the bearings of different imposed constraints, drawing more general conclusions. The method yields the high accuracy results.

2. Formulation of the optimization problem

Say F_a [N] is the force applied to the coil spring. Then the maximum torsional stress is given by the relation [10]

$$\tau = \frac{8DkF_a}{\pi d^3}\,[N/m^2], \tag{1}$$

where k is the shape coefficient of the spring that depends on the spring index. In accordance with [10], the expressions of this coefficient suggested by some authors are presented in table 1.

Table 1

The shape coefficient of the spring as a function of the spring index

Author	Göhner	Wahl	Bergsträsser	Gross	Romanian standard
Expression of k	$1+\frac{5}{4i}+\frac{7}{8i^2}+\frac{1}{i^3}$	$\frac{4i-1}{4i-4}+\frac{0.615}{i}$	$\frac{i+0.5}{i-0.75}$	$\frac{i+0.2}{i-1}$	$1+\frac{1.6}{i}$

In the specialized references the use of the shape coefficient expressions are not uniform: Göhner [5], Wahl [3,9,12], Gross[1]. In the monograph [11] the proposed expression is $1+1.5/i$, close to that of the Romanian standard.

The number of the working coils is determined by the relation

$$n_s = \frac{Gd^4}{8D^3k_a}, \tag{2}$$

where k_a $[N/m^2]$ is the spring stiffness and G $[N/m^2]$ represents the shear modulus. Knowing the suspension stiffness or derivative sag z_0 that corresponds to the automobile capacity load (see Fig.1), one can calculat the

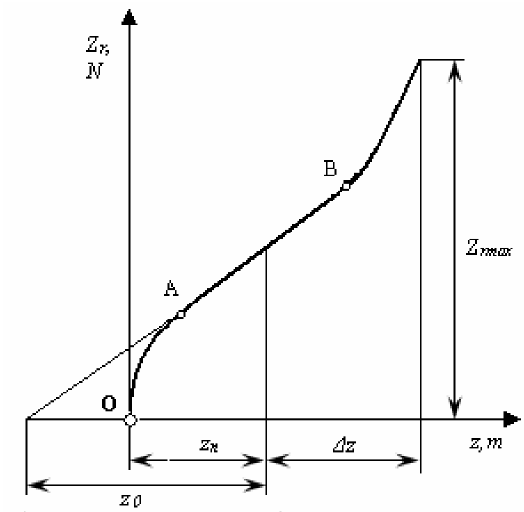

Fig.1. Characteristic elastic of suspension (corresponding to a single wheel)

stiffness of the main spring k_a or its rated sag f_n [3, 5, 7, 9] (In Fig. 1 the following notations are made: z [m]-the vertical displacement of the tyre contact area centre; z_n [m]-the sag corresponding to the capacity load; Δz [m]-the maximum variation of the wheel displacement that is limited by jounce buffer; Z_r [N], Z_{rmax} [N]-the normal reaction of the road acting on the wheel, respectively its maximum value corresponding to the maximum displacement (travel)). The variations of the spring sag that corresponds to the extreme inferior position and the superior position of the wheel are Δf_i and Δf_s, respectively. Taking into account suspension gear ratios corresponding to those positions and starting from the values of z_n and Δz one can determine Δf_i and Δf_s [5,7]. The maximum force acting on the spring is:

$$F_{a\max} = F_{an} + k_a.\Delta f_s, \tag{3}$$

and the spring force amplitude that is necessary to determine the fatigue stress is given by the following relation according to [8] :

$$\Delta F_a = 0.9(\Delta f_i + \Delta f_s)\, k_a / 2. \tag{4}$$

The allowable tangential stress under static load of the spring steel is [8]:

$$\tau_a = q\sigma_c b_0(d)/c_s, \tag{5}$$

where σ_c [N/m^2] represents the limited creep stress, q [-]-coefficient less than 1 (in general, q=0.63), b_0 [-]-coefficient depending of the wire diameter, c_s [-]-the safety factor. The allowable fatigue tangential stress is determined by the relation [5]:

$$\Delta\tau_a = q_1\sigma_r b_1(d)/c_s, \tag{6}$$

where σ_r [N/m^2] is the ultimate tensile strength of the spring steel, q_1 represents a coefficient with less than 1 values (in general, q_1=0.24) and $b_1(d)$ is a coefficient depending of the coil diameter.

The minimum length of the coil spring is [5] :

$$L_{\min} = L_b + \chi(D/d)(d + \Delta d)n_s \tag{7}$$

with

$$L_b = (n_s + 1.1)(d + \Delta d), \tag{8}$$

where Δd represents the tolerable superior deviation of the coil of nominal diameter d and χ is a coefficient used to determine the minimum gap between the spring coils when the spring is maximally compressed. The length of the coil spring under nominal load is

$$L_n = L_{\min} + \Delta f_s, \tag{9}$$

and the length of the coil spring in the free state is given by

$$L_0 = L_n + F_{an} / k_a. \tag{10}$$

From the condition that the tangential stresses corresponding to F_{amax} and ΔF_a do not exceede the allowable values we obtain the inequalities :

$$F_1(d,D) = \frac{8}{\pi d^3} Dk\left(\frac{d}{D}\right) F_{a\max} - q\sigma_c b_0(d)/c_s \leq 0, \tag{11}$$

$$F_2(d,D) = \frac{8}{\pi d^3} Dk\left(\frac{d}{D}\right) \Delta F_a - q_1\sigma_r b_1(d)/c_s \leq 0. \tag{12}$$

The coil spring should not lose the buckling stability and consequently the condition of the stability is written as:

$$F_3(n_s,d,D) = (L_0 - L_b)/L_0 - f_l(L_0/D) \leq 0, \tag{13}$$

where f_l is a function presented into a diagramm [5].

Besides the above conditions we may add the following conditions:

$$-d \leq 0,\ -D \leq 0,\ D + d - D_{e\max} \leq 0, \tag{14}$$

where D_{emax} represents the maximum outward diameter of the spring, that must not be override due to the limited space.

Finally, one takes into account that values of the spring index belong to a certain interval:

$$-i + i_{\min} \leq 0,\ i - i_{\max} \leq 0. \tag{15}$$

In the general case, one recommends i_{min}=4, i_{max}=12 [10,11] and in the case of the automobile suspension i_{min}=7 and i_{max}=12, according to [3].

The spring mass is given by the relation

$$m_a = \pi^2 (n_s + n_{sc}) D d^2 \rho / 4 [kg], \tag{16}$$

where n_{sc}=1.5÷2 represents the nonactive coil number and ρ [kg/m^3] is the density of the spring steel. Taking into account the relation (2), the preceding relation becomes

$$m_a = F_0(d, D) = f_{01}.d^6 / D^2 + f_{02}.d^2.D \tag{17}$$

with the notations

$$f_{01} = \pi^2 G \rho / (32 k_a), \quad f_{02} = \pi^2 n_{sc} \rho / 4. \tag{17'}$$

The optimization problem consists of minimizing the spring mass, namely the objective function F_0 with the following constraints: (11), (12), (13), (14) and (15).

3. Solving the problem of the spring mass minimization

The concrete approach of the optimization problem requires to know the analytical expressions of the coefficients b_0, b_1, χ and f_l. Mention is made that there are diagramms for these coefficients [5]. Starting from these diagramms and using the smallest square method we obtain the following expressions:

$$b_0 = \begin{cases} 1 \; for \; d \le 0.010m \\ 100d^2 - 8.5d + 1.0766 \; for \; d > 0.010m \end{cases} \tag{18}$$

$$b_1 = \begin{cases} 1 \; for \; d \le 0.010m \\ 100d^2 - 5.8d + 1.069 \; for \; d > 0.010m \end{cases} \tag{19}$$

$$\chi = 10^{-4}.i^3 - 2.4.10^{-3}.i^2 + 0.0264i + 0.0166 \tag{20}$$

$$f_l = -0.01(L_0 / D)^2 + 0.007(L_0 / D) + 0.7338. \tag{21}$$

As it has been found , the constraints number of the optimization problem is 8, consequently it's large enough. In view of the present automobile construction it is possible to start with a more reduced constraints number. One solves the optimization problem and verifies if the other constraints are satisfied thereafter. If the constraints are not satisfied the problem is resumed with the additional constraints.

By examining the conditions (11) and (12) we observe that these are of similar structure. It is possible that if one of the conditions is satisfied the other be also satisfied. If, for example, the static strength condition (11) is more severe than the fatigue strength condition (12), then by comparing of these inequalities we obtain:

$$100(\varphi-1)d^2-(8.5\varphi-5.8)d+1.0766\varphi-1.069\le 0, \qquad (22)$$

where

$$\varphi=\frac{q}{q_1}\cdot\frac{\sigma_c}{\sigma_r}\cdot\frac{\Delta F_a}{F_{a\max}}. \qquad (23)$$

The variation interval of d that satifies the inequality (22) is easilly to find. In this case we use the condition (11) only. If d does not belong to this interval, we use the constraint (12) only.

If we consider the constraints (11) and (14), then the Lagrange's function may be written

$$\ell=F_0(d,D)+\lambda_1F_1(d,D)+\lambda_2(D+d-D_{e\max}+\lambda_3d+\lambda_4D, \qquad (24)$$

where λ_j ($j\in\{1,2,3,4\}$ are the Lagrange's multipliers. The condition of Kuhn-Tucker is expressed as [13]:

$$\begin{gathered}\frac{\partial\ell}{\partial d}=0,\frac{\partial\ell}{\partial D}=0,\lambda_j\ge 0(j\in\{1,2,3,4\}),\lambda_1F_1=0,\\ \lambda_2(D+d-D_{e\max})=0,\lambda_3d=0,\lambda_4D=0.\end{gathered} \qquad (25)$$

Using the Göhner's expression of the shape coefficient we obtain the following relations of the partial derivatives

$$\begin{aligned}&\frac{\partial\ell}{\partial d}=6f_{01}\frac{d^5}{D^2}+2f_{02}dD-[(\frac{24}{\pi}\frac{D}{d^4}+\frac{20}{\pi}\frac{1}{d^3}+\frac{7}{\pi}\frac{1}{d^2D})F_{a\max}+\\ &\frac{q\sigma_c}{c_s}(200d-8.5)]\lambda_1+\lambda_2+\lambda_3=0, \qquad (26)\\ &\frac{\partial\ell}{\partial D}=-2f_{01}\frac{d^6}{D^3}+f_{02}d^2+(\frac{8}{\pi}\frac{1}{d^3}-\frac{7}{\pi}\frac{1}{dD^2}-\frac{16}{\pi}\frac{1}{D^3})F_{a\max}\lambda_1+\lambda_2+\lambda_4=0.\end{aligned}$$

If $d \le 0.01$m, then the second term of the square bracket of the first relation (26) is equal to zero.

If $\lambda_3>0$ and $\lambda_4>0$ then $d=0$ and $D=0$. Therefore, the results for the function F_0 are absurd. Consequently, $\lambda_3=0$ and $\lambda_4=0$. If $\lambda_1=0$ and $\lambda_2=0$, from (26) we obtain

$$3f_{01}\frac{d^5}{D^2}+f_{02}dD=0,-2f_{01}\frac{d^6}{D^3}+f_{02}d^2=0, \tag{27}$$

that could not be simultaneously satisfied if $d>0$, $D>0$. As a result, the objective function does not have a local minimum in the first quadrant of the coordinate system and the optimum point belongs to the border of the admissible domain.

Finally, we will examine the following conditions: a) $\lambda_1=0$, $\lambda_2>0$; b) $\lambda_1>0$, $\lambda_2=0$; c) $\lambda_1>0$, $\lambda_2>0$. If the condition a) is satisfied it is easy to observe that the first relation of (26) could not be fulfilled, so that the cases b) and c) are only considered. Therefore, the optimum point cannot exclusively belong to the border given by the third relation (14). Consequently, a necessary condition for the minimum of the spring mass is $F_1(d, D)=0$, namely the stress should have the allowable prescribed value. In the case b) there are three equations and three unknown quantities: d, D and λ_1. Eliminating λ_1 between relations (26) we obtain (it is assumed that $d>0.010m$):

$$\left[(24D^3+20dD^2+7d^2D)F_{a\max}+\pi\frac{q\sigma_c}{c_s}(200d-8.5)d^4D^2\right](2f_{01}d^9-f_{02}d^5D^3)$$
$$-(8D^3-7d^2D-16d^3)(16f_{01}d^9+2f_{02}d^5D^3)F_{a\max}=0. \tag{28}$$

If $d\le 0.010m$ then the second term of the square bracket is zero. The nonlinear system of the equations (28) and $F_1(d, D)=0$ may be solved by known numerical methods. So the solution of the optimization problem is obtained. In the case c) the conditions appear:

$$F_1(d,D)=0,\ D+d-D_{e\max}=0. \tag{29}$$

The solution of the problem is the numerical solution of the system (29).

It stands to reason that in a certain specific case one of the two situations b) or c) could only appear. In order to establish what situation is really involved, the two systems are separately solved determining the values of the objective function (the spring mass). The solution of the optimization problem corresponds to the least value of the objective functions. Of course, in the same time, one verifies if the constraints are fulfilled in accordance with the given situation. After that, one can pass to the check up the constraints (13) and (15).

In a similar way to that above presented, it is proved that the optimum point does not exclusively belong to the border corresponding to the minimum value of the spring index (see first condition (15)). This point could even be the intersection point of the border given by (11) and the first relation of (15). Its coordinate are obtained by the solving the system of the equations that correspond to two borders. In general, one can say nothing regarding the fact that the optimum point belongs or not to the border corresponding to the maximum spring index exclusively. If after the equations are solved according to the situation b) and c) it is found that the second constraint of (15) is not fulfilled, then we consider the system of the equations corresponding to the equalities of (11) and second condition of (15). Its solving leads to the optimum solution.

When the constraints (13) are not fulfilled, the Lagrange's function is properly completed. After that a nonlinear system should be solved, that requires knowing the expressions of the partial derivatives of the function F_3 with respect to d and D. These expressions are enough intricate so that it is recommended to approach the problem differently, namely in connection with the plotting of the constraints. In the present case, when there are two variables d and D, the plotting is easilly to make, being very suggestive. In a concrete case it makes evident the above mentioned situations and the problem solving is easier.

4. Working optimization example

In [5] it is minutely presented a work example of the calculation of a suspension helical spring of a car front axle. In order to make a comparison, in the present exemplification the initial necessary data for calculation of the spring are taken over from the above mentioned work example. The specifications are the followings: F_{an}=4400N, ΔF_a=2600N, k_a=42800N/m, σ_r=1600MPa, σ_c=1450MPa, q=0.63, q_1=0.24, G=80000MPa, ρ=7800kg/m^3, c_s=1.1, Δd=0.08mm, n_{sc}=1.5. We consider further: i_{min}=7, i_{max}=12 and D_{ex}=200mm.

The plotting of the borders given by constraints (14) and (15) are achieved without intricacy. Relating to the constraints (11), (12) and (13), in MATLAB the functions F_1, F_2 and F_3 are defined and using function **Contour** the zero level lines are plotted. The obtained results are shown in fig. 2. In this Figure the conditions corresponding to the shown curves are specified. It is found that, in the present case, the constraint relating to the stress corresponding to the maximum load of the spring is more severe than the constraint in connection with the fatigue stress. The condition of buckling avoidance defines a domain that is included in the feasible fatigue domain, but not included in the feasible domain of the maximum load. So we obtain the intersecting points A and B of the borders corresponding to the maximum stress and the buckling. It is to observe that these points are situated in the feasible domain defined by the maximum value the minimum value of the spring index. Therefore, at least in the present case, the spring index is not drawn near to the mentioned values, particularly to the minimum value. In exchange, the condition regarding the maximum spring diameter is restrictive and the point C (see Fig. 2) is located between the points A and B. As it has been shown, the optimum point can't be situated on the straight line corresponding to the maximum diameter. Therefore it belongs to the border CAF.

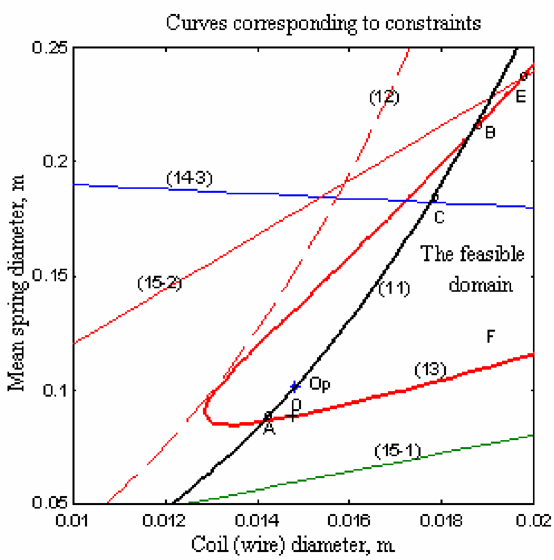

Fig.2. The borders of the constraints and the feasible domain

Solving the system of the Equations (28) and $F_1(d, D)$ =0, we get the solution: d=14.81*mm* and D=100.98*mm*. This solution defines the optimum point O_p (Fig. 2), located on the border AC. In this way, in the case that has been examined here, the optimization problem has been solved.

To clearly represent the optimum conditions, in Figure 3 the level lines of the objective function are shown. On the level lines the values of the spring mass, inclusively the minimum value (5.298*kg*) are inscribed.

Also, on the same figure one shows and the border relating to the maximum loading (the other borders are not shown). In general, it can proved that the solution of the mentioned system corresponds to the point of tangency between a level line and the border defined by F_1=0. The representation on the figure confirms this affirmation. Following the position and the aspect of the level lines one finds that the optimum point can't belong to the border AF, so that the solution of the problem is that mentioned above. In further specific cases, the representation of the same kind allows to make evident the border on which the optimum point can be situated. It is necessary to observe that, according to the location of the level lines, in the feasible domain the sprung mass increases in proportion as the figurative point moves away from the constraint curve for the maximum stress. In spite of all these, there are points inside of the domain where the spring mass is less than the mass corresponding to the same points situated on the constrained curve. Therefore, the calculation corresponding to the maximum load does not always lead to the reduced mass.

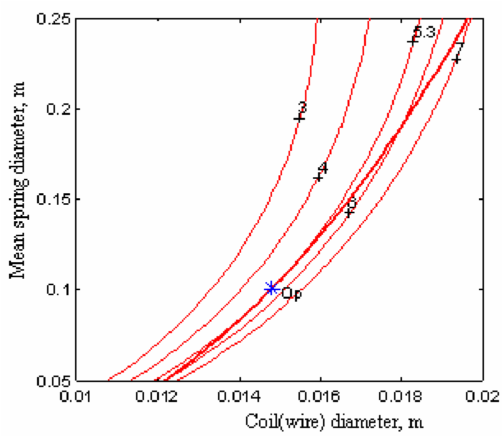

Fig.3. Level lines of the objective function and the string mass

The optimization problem may be also solved in a direct way using TOOL BOX-Optimization from MATLAB. In the present case, using this mean, we get a result similar to the above mentioned result, namely: d=14.81*mm*, D=100.98*mm,* m_a=5.543*kg,* n_s=5.39. Therefore, in comparison with the optimum solution, in [8] there is a mass increase of 4.6%. It is noticed that there are large enough differences for the spring diameter and the working coil number. It is interesting to investigate the variation of the spring parameters as a function of the coil (wire) diameter when the calculation is made for the condition of the maximum loading (for a given spring stiffness). With that in view, considering d as a parameter we solve equation $F_1(d, D)=0$ using a adequate program in MATLAB (we consider $d\in[d_A, d_B]$, where d_A and d_B are the values of the coil diameters corresponding to the points A and B at the buckling limit). The obtained results are concentrated in the Figs. 4, 5, 6 and 7.

The relative variation of the spring mass in comparison with minimum mass is plotted in Figure 4. It is found that in the case maximum coil diameter we get a mass increase of 16% (the values nearer to d_A ensure a reduced mass). Together with that, the spring diameter has an important increase.

The recommendation of [3] with regard to the choice of the spring diameter as large as possible leads to a signifiant mass increase, even if spring length decreases. For the diameters nearer to the optimum value the working coil number is great. This decreases considerably for the large coil diameters. Taking into account possible design constraints relating to the spring diameter and to the lengths the mass variation in comparison with the minimum value by means of the Fig. 4 may be estimated.

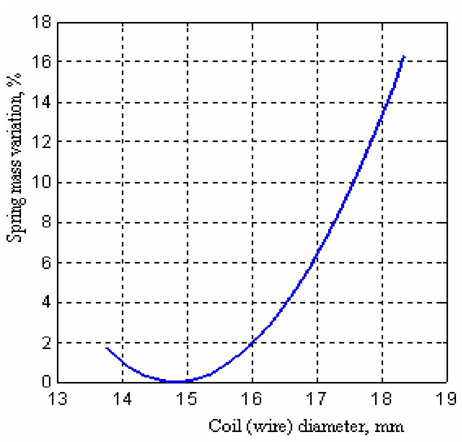

Fig.4. The relative variation of the spring mass as a function of the wire diameter

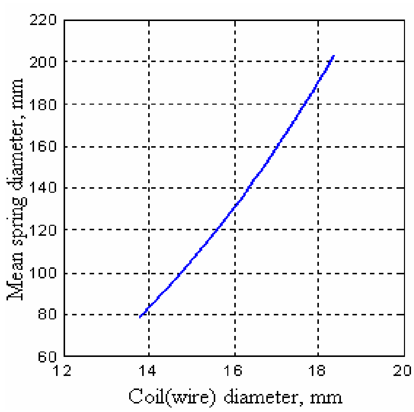

Fig.5. The spring diameter as a function of the wire diameter

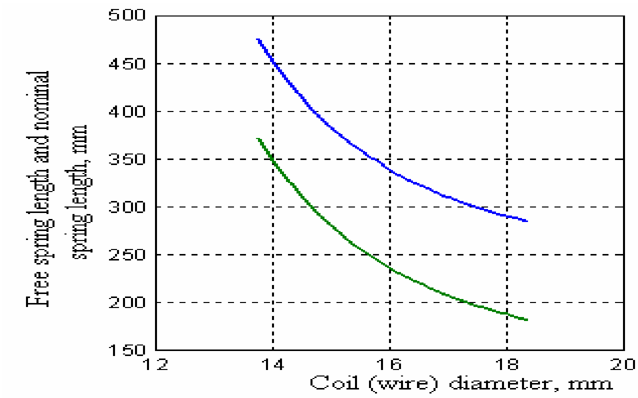

Fig.6. The free length and the nominal spring length as a functions of the coil diameter

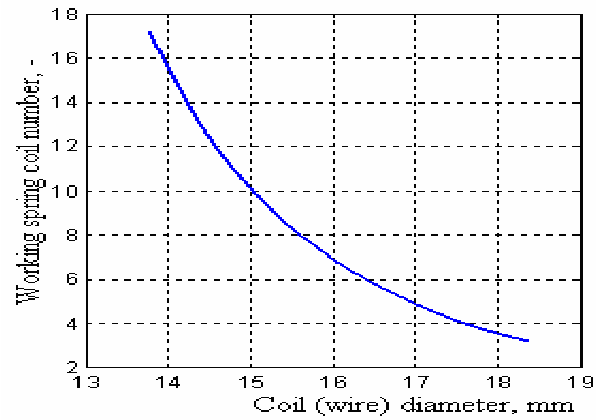

Fig.7. The working spring coil number as a function of the wire diameter

Finally, it is to observe that in another concrete case the relative placing of the points B, C and E may be different from that of the present example. So we should consider the corresponding situations that have been analysed in section 3 and the optimization problem can be completely solved.

5. Conclusions

Expressing analytically the coefficients that are necessary to calculate certain helical spring stresses of an automobile suspension we have elaborated a nonlinear programming model with constraints for the optimal design of an automobile spring suspension according to the criterion of the minimum mass.

The optimization method is associated with the solving of certain algebraical systems by means of usual computer programs. These systems are minutely presented in the paper.

The reduction of the spring mass by optimal design may be of 16%.

The choice of a large spring diameter as one recommends sometimes in literature has as a result large value of the spring mass.

R E F E R E N C E S

[1]. *D. Bastow,G. Howard,* Car Suspension and Handling, Pentech Press London, Society of Automotive Engineers, Inc., Warrendalle, 1993

[2]. *I.Draghici et al.,* Suspensii şi amortizoare, Editura Tehnică, Bucureşti, 1970

[3]. *A. I. Grishkevich* (red), Avtomobili-Konstruktsya, Konstruirovanie i Raschet. Sistemy upravlenyea i hodovaia cheasty, Vysheishaya Shkola, Minsk, 1987

[4]. *P. Lukin,G. Gasparyants,V. Rodionov,.* Automobile Chassis-Design and Calculations, MIR Publishers, Moscow, 1989

[5]. *J.Reimpell,* Fahrwerktechnik : Federung, Fahrwerkmechanik, zweite Auflage, translated into Russian, Mashinostroenie, Moskva, 1987

[6]. *J. Reimpell &H. Stoll,* The Automotive Chassis: Engineering Principles, Society of Automotive Engineers, Inc., Warrendale, 1996

[7]. *A.P.Stoicescu,* Dinamica autovehiculelor, vol. III, Institutul Politehnic Bucureşti, 1986

[8]. *M.Untaru et al.,* Calculul şi construcţia automobilelor, Editura Didactică şi Pedagogică, Bucureşti, 1982

[9]. *I.N.Uspensky, A. A. Melnikov.* Proektirovanie podveski avtomobilea, Mashinostroenie, Moskva, 1976

[10]. *Gh. Manea,.* Organe de Maşini, vol. I, Editura Tehnică, Bucureşti, 1970

[[11]. *S. D. Ponomarev, L. E. Andreeva,* Raschet uprugih elementov mashin i priborov, *Mashinostroenie*, Moskva, 1980

12]. *D.J. Wilde,* Globally Optimal Design, A Wiley-Interscience Publication John Willez & Sons, translated into Russian, Mir, Moskva, 1981

[13]. *E. J. Haugh, J.S. Arora,* Applied Optimal Design-Mechanical and Strucural Systems. A Wiley-Interscience Publication John Wiley & Sons , translated into Russian, Mir, Moskva, 1983

[14]. *** Spring Design Manual, Society of Automotive Engineers, Inc., Warrendale, 1989