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COUNTEREXAMPLES TO THE COMPLEX VERSION OF EHRESMANN'S FIBRATION THEOREM

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In this paper we give a counterexample to the following problem: if $\pi : M \to N$ is a holomorphic map between complex manifolds such that M is Stein and the fibers are biholomorphic to each other, then π is a locally trivial fibration.

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1. Introduction

One fundamental result in differential topology is Ehresmann's Fibration Theorem [1] which states that a smooth proper submersion between diffe-rentiable manifolds is a locally trivial fibration.

Unfortunately, Ehresmann's Theorem has no analytic analogue and counterexamples were first constructed by Kodaira and Spencer [6]. They gave a necessary and sufficient condition for a holomorphic family with compact fibers to be locally trivial. Other counterexamples can be found in [4].

Another necessary and sufficient condition for a holomorphic family with compact fibers to be locally trivial is the following: a holomorphic family (M, π, N) such that π is proper is locally trivial if and only if the fibers $\pi^{-1}(y)$ for every $y \in N$ are biholomorphic to each other. The direct implication was conjectured by Stein and proved by Grauert [4] and the converse is due to Fischer and Grauert [2]. The key element of the previous result is that the fibers are compact.

In the following we will present an example dealing with the case of non-compact fibers.

2. Preliminaries

Definition 2.1. Let $\pi : M \to N$ be a smooth map between differentiable manifolds. We say that π is a *submersion* if π is surjective and for every $x \in M$ the differential map $d_x\pi : T_xM \to T_{\pi(x)}N$ is surjective.

It is obvious that the projection $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ onto the first n coordinates is a submersion.

Definition 2.2. Let X and Y locally compact topological spaces. A continuous mapping $f : X \to Y$ is called *proper* if for every compact set K in Y the preimage $f^{-1}(K)$ is compact.

In order to make things simpler and write more concise we give the following definition.

Definition 2.3. Let $\pi : M \to N$ be a holomorphic map between complex ma-nifolds. If π is a submersion, we say that the triple (M, π, N) is a *holomorphic family.*

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Definition 2.4. We say that a holomorphic family (M, π, N) is *locally trivial* (or that π is a *locally trivial fibration*) if for every $y \in N$ there exists an open neighbourhood U of y and a biholomorphic map $f : \pi^{-1}(U) \to U \times \pi^{-1}(y)$ such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad f \quad} & U \times \pi^{-1}(y) \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle p} & \\ & U & \end{array}$$

is commutative, where p is the projection of the Cartesian product on the first factor.

Let Ω be an open subset of $\mathbb{C}^n$ and $f : \Omega \to \mathbb{R}$ a smooth function (for simplicity, "smooth" will stand for $\mathcal{C}^\infty$).

Definition 2.5. If $z_0 \in \Omega$, then the *Levi form of* f *at* z_0, denoted by $L(f, z_0)$, is the quadratic form determined by

$$\left(\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}(z_0)\right)_{i,j}.$$

A function f is called *strictly plurisubharmonic* if its Levi form is positive definite at any point of Ω.

Consider now Ω to be a smoothly bounded domain in $\mathbb{C}^n$. Let $x \in \partial\Omega$ and let U be an open neighborhood of x in $\mathbb{C}^n$.

A smooth real-valued function ϕ on U is called a *defining function on* U *for* Ω if the following conditions hold:

(1) $\Omega \cap U = \{\phi < 0\}$;
(2) $d\phi \neq 0$ on $\partial\Omega \cap U$.

Definition 2.6. If the restriction of the Levi form $L(\phi, z)$ to the tangent space $T_p(\partial\Omega)$ is positive definite, then Ω is said to be *strictly pseudoconvex at* p. Finally, Ω is said to be *strictly pseudoconvex* if it is strictly pseudoconvex at each boundary point.

The next observation will turn out to be a key argument in the construction of the counterexample to Ehresmann's Fibration Theorem.

Remark 2.1 ([3], pp. 65-66)**.** *The definition of a* Ω *being strictly pseudoconvex at a point* p *does not depend on the choice of the boundary function and the fact that the Levi form of the defining function should be positive definite is invariant under biholomorphic transformations.*

Definition 2.7. A domain $\Omega \subset \mathbb{C}^n$ is called *circular* if $e^{i\theta} z \in \Omega$ for every $z \in \Omega$ and $\theta \in \mathbb{R}$.

Circular domains are particular cases of the so called *Reinhardt domains* (for the precise definition check [7]). One important remark is that the domains of convergence of power series are Reinhardt domains.

Definition 2.8. A domain $\Omega \subset \mathbb{C}^n$ is called *homogenous* if, for each pair of points $z, w \in \mathbb{C}^n$, there exists $f \in \mathrm{Aut}(\Omega)$ such that $f(z) = w$.

The classical Riemann Mapping Theorem says that every proper, simply connected open subset of $\mathbb{C}$ is biholomorphic to the disc, or any two simply connected domains in $\mathbb{C}$, which are not all of $\mathbb{C}$, are biholomorphic to each other. There is no analogue of this result for domains in $\mathbb{C}^n$, $n > 1$.

Historically, the first result that supports the previous statement was proved by Poincaré: the unit ball and the unit polydisc in $\mathbb{C}^n$, $n > 1$ are not biholomorphic. Various proofs can be found in the literature, but the one which is important for our further purposes is the one in [5] (pp. 9-10). This relies on the Schwarz Lemma (the one-dimensional version) and the Chain Rule. We are also interested in the following result.

Proposition 2.1 ([5], pp. 16-17)**.** *If $f : G \to H$ is a biholomorphic mapping between bounded circular domains, and if $0 \in G$ and $f(0) = 0$, then f is linear.*

3. **The Example**

In the following we will give an example of a holomorphic family (M, π, N) such that M is Stein and the fibers are biholomorphic to each other, but π is not a locally trivial fibration. Also it will turn out that this essentially happens because π is not proper.

Example 3.1. Consider

$$M = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < 1\} \subset \mathbb{C}^2$$

and

$$N = \{z \in \mathbb{C} : |z|^2 < 1\} \subset \mathbb{C}.$$

Let $\pi : M \to N$ be the projection on the first coordinate: $\pi(z, w) = z$.

Proof. It is obvious that M is the unit ball in $\mathbb{C}^2$ and N is the unit disc in $\mathbb{C}$. Also it is clear that M is Stein and that (M, π, N) is a holomorphic family.

Now we are interested in determining the fibers. If $z \in N$, then $\pi^{-1}(z) = \{z\} \times D_z$, where D_z is the disc centered at the origin with radius $1-|z|^2$. So the fibers are biholomorphic to a disc.

Next we show that π is not a locally trivial fibration. Assume the contrary. Consider $z \in N$; then there exists U an open neighbourhood of z and a biholomorphic map $f : \pi^{-1}(U) \to U \times \pi^{-1}(z)$ such that the diagram in Definition 2.4 is commutative. We take $z = 0$, and we have that there exists a biholomorphism

$$F : D(0, r) \times D \to \pi^{-1}(U),$$

where $D(0, r)$ is the disc centered at 0 with radius $r < 1$ and D is the unit disc (both in $\mathbb{C}$). We note that the domain of F is a polydisc

$$P = \{(z, w) \in \mathbb{C}^2 : |z| < r, |w| < 1\},$$

and the codomain is

$$\pi^{-1}(U) = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < 1, |z| < r\}.$$

It is obvious that both the domain and the codomain of F are bounded circular domains. Since P is a homogenous domain (see [5], pp. 9-10), using a well chosen automorphism of P, we may assume that $F(0) = 0$. This implies, by Proposition 2.1, that F is a linear biholomorphism. Thus F can be extended across the boundary of P and we get that $F(\partial P) = \partial(\pi^{-1}(U))$, which is equivalent to $\partial P = F^{-1}(\partial(\pi^{-1}(U)))$.

Now we are trying to involve the boundaries of P and $\pi^{-1}(U)$. We have that

$$\begin{aligned}\partial P =& \{(z, w) \in \mathbb{C}^2 : |z| < r, |w| = 1\}\\ &\cup \{(z, w) \in \mathbb{C}^2 : |z| = r, |w| < 1\}\\ &\cup \{(z, w) \in \mathbb{C}^2 : |z| = r, |w| = 1\}\end{aligned}$$

and

$$\begin{aligned}\partial(\pi^{-1}(U)) =& \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < 1, |z| = r\}\\ &\cup \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1, |z| < r\}\\ &\cup \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1, |z| = r\}.\end{aligned}$$

Let $p \in \partial(\pi^{-1}(U))$ such that $p \in \{|z|^2 + |w|^2 = 1, |z| < r\}$. Then there exists a sufficiently small open ball V centered in p so that $V \cap \partial(\pi^{-1}(U)) \subset \{|z|^2 + |w|^2 = 1, |z| < r\}$.

We define $\varphi : V \to \mathbb{R}$ by $\varphi(z, w) = |z|^2 + |w|^2 - 1$.

We have that φ is a defining function on V for $\pi^{-1}(U)$ and its Levi form is positive definite on the tangent space of $\pi^{-1}(U)$ at p (the eigenvalue is 1). We obtain that $\varphi \circ F : F^{-1}(V) \to \mathbb{R}$ is a strictly plurisubharmonic, defining function on $F^{-1}(V)$ for P. Since $F^{-1}(p) \in \partial P$, we deduce that P is strictly pseudoconvex at $F^{-1}(p)$. But $F^{-1}(p)$ is a point on the smooth boundary of P and, without loss of generality, we can assume that $F^{-1}(p) \in D(0, r) \times \partial D$. Define $\psi : F^{-1}(V) \to \mathbb{R}$ by

$$\psi(z, w) = |w|^2 - 1.$$

We hava that $F^{-1}(V)$ is a neighborhood of $F^{-1}(p)$. Thus ψ is a defining function on $F^{-1}(V)$ for P, but its Levi form is not positive definite on the restriction to the tangent space $T_{F^{-1}(p)}(\partial P)$ (the eigenvalue is 0).

So we have obtained two defining functions for P at $F^{-1}(p)$ such that the Levi form of one of them, namely $\varphi \circ F$, is positive definite on the tangent space $T_{F^{-1}(p)}(\partial P)$ and the Levi form of the other, namely ψ, is not. This contradicts Remark 2.1 and the proof is done. □

As mentioned before, it is easy to observe that π is not proper.

4. Conclusions

Ehresmann's Fibration Theorem has no analytic analogue and the fact that the fibers need to be biholomorphic to each other is a necessary condition in both cases of compact and non-compact fibers. In the case of non-compact fibers this is not a sufficient condition. Having in mind the ideas that are used in one of the proofs of Poincaré's Theorem (the unit ball and the unit polydisc are not biholomorphic for $n > 1$) we are able to construct a counterexample that supports our previous assertion.

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