ON GENERALIZED ABSOLUTE VALUE EQUATIONS

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In this paper, we consider the generalized absolute value equations. It is shown that Lax-Milgram lemma and absolute values equations can be obtained as special cases. We use the auxiliary principle technique to prove the existence of a solution to the generalized absolute value equations. This technique is also used to suggest some new iterative methods for solving the generalized absolute value equations. The convergence analysis of the proposed methods is analyzed under some mild conditions. Ideas and techniques of this paper may stimulate further research.

Keywords: Absolute value equations, Lax-Milgram Lemma, Auxiliary Principle, Iterative method, Convergence.

1. Introduction

Recently much attention has been given to solve the systems of absolute value equations, which were introduced and studied by Mangasarian and Meyer [12]. The system of absolute value equations are closely related to the complementarity problems, variational inequalities and optimization problem. Various numerical methods have been developed for solving the absolute values equations, see [5, 6, 9, 10, 11, 12, 13, 14, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32] and the references therein.

In this paper, we introduce and study the system of generalized absolute values equations. This system of generalized absolute value equations can be viewed as weak formulation of the boundary value problems. It is shown that the system of absolute value equations [12] and Lax-Milgram [7] can be obtained as special cases. We use the auxiliary principle technique, which is mainly due to Lions and Stampachhia [8] and Glowinski et al [4], to discuss the existence to a solution of the generalized absolute value equations. The auxiliary principle technique is used to suggest some iterative methods for solving the generalized absolute value equations.

In Section 2, we introduce the generalized absolute value equations and discuss their applications. The auxiliary principle technique is used to discuss the existence to a solution as well as to suggest some iterative methods for solving the general absolute value equations. The convergence analysis of the proposed method is also considered under some mild conditions. As special cases, two new iterative methods for solving the absolute values equations are obtained.

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2. **Formulations and basic facts**

Let H be a Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively.

For a given an operator L, continuous functional f, and a constant λ, we consider the problem of finding $u \in H$, such that

$$\langle Lu - \lambda|u|, v - u\rangle = \langle f, v - u\rangle, \quad \forall v \in H, \tag{1}$$

which is called the system of generalized absolute value equations. Here $|u|$ denotes the component-wise absolute value of $u \in H$. A wide class of problems arising in pure and applied sciences can be studied via the absolute valued equations (1).
The problem (1) is equivalent to finding $u \in H$, such that

$$\langle Lu - \lambda|u|, v\rangle = \langle f, v\rangle, \quad \forall v \in H. \tag{2}$$

If $\lambda = 0$ and $\langle Lu, v\rangle = a(.,.)$, where $a(.,.) : H \times H \to H$, is a continuous bifunction, then the problem (2) is equivalent to finding $u \in H$, such that

$$a(u, v) = \langle f, v\rangle, \quad \forall v \in H, \tag{3}$$

which is known as the famous Lax-Milgram Lemma[7]. This result has been used to discuss the unique existence to a solution of the boundary value problems. This result is of similar significance in the study of function spaces and partial differential equations. For the applications and generalizations of the Lax-Milgram Lemma, see [1, 3, 7, 8, 15, 18, 22] and the references.

We note that the problem (1) is equivalent to finding $u \in H$ such that

$$Lu - \lambda|u| = f, \tag{4}$$

which is known as the absolute value equations. Problem of type (1) has been discussed in a series of papers recently, see[5, 6, 9, 10, 11, 12, 13, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32].
We would like to mention that problems (1), (2) and (4) are equivalent.

If the operator L is linear, positive and symmetric, then problem (1) is equivalent to finding a minimum of the function$I[v]$ on H, where

$$I[v] = \langle Lv - \lambda|v|, v\rangle - 2\langle f, v\rangle \quad \forall v \in H, \tag{5}$$

which is a nonlinear quadratic programming problem and can be solved using the known techniques of the nonlinear optimization.

We now show that the boundary value problems can be studied via problem (1).

Example 2.1. Consider the absolute boundary value problem of finding u such that

$$\frac{D^2u}{dx^2} - \lambda|u| = f(x), \quad \forall x \in [a, b], \tag{6}$$

with boundary conditions

$$u(a) = 0, \quad u(b) = 0, \tag{7}$$

where $f(x)$ is a continuous function. This problem can be studied in the general framework of the problem (1). To do so, let $H_0^1[a, b] = \{u \in H, u(a) = 0, u(b) = 0\}$ be a Hilbert space, see [4]. One can easily show that the energy functional associated with (1) is:

$$\begin{aligned} I[v] &= -\int_a^b \frac{d^2v}{dx^2} v dx + \int_a^b \lambda |v| v dx - 2\int_a^b f v dx, \quad \forall u \in H_0^1[a,b] \\ &= \int_a^b (\frac{dv}{dx})^2 + \int_a^b \lambda |v| v dv - 2\int_a^b f v dx \\ &= \langle Lv, v\rangle + \langle \lambda |v|, v\rangle - 2\langle f, v\rangle. \end{aligned}$$

where

$$\langle Lu, v\rangle = \int_a^b \frac{du}{dx}\frac{dv}{dx} dx, \tag{8}$$

and

$$\langle |u|, v\rangle = \int_a^b |u| v dx, \quad \langle f, v\rangle = \int_a^b f v dx.$$

It is clear that the operator L defined by (8) is linear, symmetric and positive.
Thus the minimum of the functional $I[v]$ defined on the Hilbert space $H_0^1[a,b]$ can be characterized by equation (2) or equivalently (1).This shows that the absolute boundary value problems can be studied in the framework of (1).

We now recall some basic concepts and results.

Definition 2.1. *An operator $L : H \longrightarrow H$ is said to be;*
(i). Strongly monotone, if there exists a constant $\alpha > 0$, such that

$$\langle Lu - Lv, u - v\rangle \geq \alpha \|u - v\|^2, \quad \forall u, v \in H.$$

(ii). Lipschitz continuous, if there exists a constant $\beta > 0$, such that

$$\|Lu - Lv\| \leq \beta \|u - v\|, \quad \forall u, v \in H.$$

(iii) monotone, if

$$\langle Lu - Lv, u - v\rangle \geq 0, \quad \forall u, v \in H.$$

We remark that, if the operator L is both strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, respectively, then from (i) and (ii), it follows that $\alpha \leq \beta$.

3. Main results

In this section, we use the auxiliary principle technique, the origin of which can be traced back to Lions and Stampacchia [7]and Glowinski et al [4], as developed by Noor [15, 16, 18]. The main of idea of this technique to consider an auxiliary problem related to the original problem. This way, one defines a mapping connecting the solutions of both problems. To prove the existence of solution of the original problem, it is enough to show that this connecting mapping is a contraction mapping and consequently has a unique solution of the original problem. Another novel feature of this approach is that this technique enables us to suggest some iterative methods for solving the generalized absolute value equations.

Theorem 3.1. *Let the operator L be strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, respectively. If there exists a constant $\rho > 0$ such that*

$$0 < \rho < \frac{2(\alpha - \lambda)}{\beta^2}, \quad \rho\lambda < 1, \quad \lambda < \alpha. \tag{9}$$

then problem (1) has a unique solution..

Proof. (a). **Uniqueness**. Let $u_1 \neq u_2 \in H$ be two solutions of problem (1). Then

$$\langle Lu_1 - \lambda|u_1|, v - u_1\rangle = \langle f, v - u_1\rangle, \quad \forall v \in H. \tag{10}$$

$$\langle Lu_2 - \lambda|u_2|, v - u_2\rangle = \langle f, v - u_2\rangle, \quad \forall v \in H. \tag{11}$$

Taking $v = u_2$ in (10) and $v = u_1$ in (11) and adding the resultant, we have

$$\langle Lu_1 - Lu_2 - \lambda(|u_1| - |u_2|), u_1 - u_2\rangle = 0. \tag{12}$$

Using the strongly monotonicity of L with constant $\alpha > 0$ and (12), we have

$$\begin{aligned} \alpha\|u_1 - u_2\|^2 &\leq \langle Lu_1 - Lu_2, u_1 - u_2\rangle \\ &= \langle \lambda(|u_1| - |u_2|), u_1 - u_2\rangle \\ &\leq \lambda\||u_1| - |u_2|\|\|u_1 - u_2\| \\ &\leq \lambda\|u_1 - u_2\|^2, \end{aligned}$$

from which, it follows that

$$(\alpha - \lambda)\|u_1 - u_2\|^2 \leq 0,$$

which implies that $u_1 = u_2$, the uniqueness.
(b). **Existence.** We now use the auxiliary principle technique to prove the existence of a solution of (1). For a given $u \in H$ satisfying (1), consider the problem of finding $w \in H$ such that,

$$\langle \rho(Lu - \lambda|u|), v - w\rangle + \langle w - u, v - w\rangle = \rho\langle f, v - w\rangle, \quad \forall v \in H, \tag{13}$$

which is called the auxiliary problem, where $\rho > 0$ is a constant. It is clear that (13) defines a mapping w connecting the both problems (1) and (13). To prove the existence of a solution of (1), it is enough to show that the mapping w defined by (13) is a contraction mapping.

Let $w_1 \neq w_2 \in H$ (corresponding to $u_1 \neq u_2$) be solutions of (13). Then

$$\langle \rho(Lu_1 - \lambda|u_1|), v - w_1\rangle + \langle w_1 - u_1, v - w_1\rangle = \langle f, v - w_1\rangle, \quad \forall v \in H, \tag{14}$$

$$\langle \rho(Lu_2 - \lambda|u_2|), v - w_2\rangle + \langle w_2 - u_2, v - w_2\rangle = \langle f, v - w_2\rangle, \quad \forall v \in H. \tag{15}$$

Taking $v = w_2$ in (14) and $v = w_1$ in (15) and adding the resultant, we have

$$\begin{aligned} \|w_1 - w_2\|^2 &= \langle w_1 - w_2, w_1 - w_2\rangle \\ &= \langle u_1 - u_2 - \rho(Lu_1 - Lu_2) + \rho\lambda(|u_1| - |u_2|), w_1 - w_2\rangle. \end{aligned} \tag{16}$$

From (16), we have

$$\|w_1 - w_2\|^2 \leq \|u_1 - u_2 - \rho(Lu_1 - Lu_2) + \rho\lambda(|u_1| - |u_2|)\|\|w_1 - w_2\|$$

from which, it follows that

$$\begin{aligned} \|w_1 - w_2\| &\leq \|u_1 - u_2 - \rho(Lu_1 - Lu_2)\| + \rho\lambda\||u_1| - |u_2|\| \\ &\leq \|u_1 - u_2 - \rho(Lu_1 - Lu_2)\| + \rho\lambda\|u_1 - u_2\|. \end{aligned} \tag{17}$$

Using the strongly monotonicity and Lipschitz continuity of the operator L with constants $\alpha > 0$ and $\beta > 0$, we have

$$\begin{aligned} \|u_1 - u_2 - \rho(Lu_1 - Lu_2)\|^2 &= \langle u_1 - u_2 - \rho(Lu_1 - Lu_2), u_1 - u_2 - \rho(Lu_1 - Lu_2)\rangle \\ &= \langle u_1 - u_2, u_1 - u_2\rangle - 2\rho\langle Lu_1 - Lu_2, u_1 - u_2\rangle \\ &\quad + \rho^2\langle Lu_1 - Lu_2, Lu_1 - Lu_2\rangle \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u_1 - u_2\|^2. \end{aligned} \tag{18}$$

Combining (17) and (19), we have

$$\begin{aligned}\|w_1 - w_2\| &\le \big(\rho\lambda + \sqrt{(1 - 2\rho\alpha + \rho^2\beta^2)}\big)\|u_1 - u_2\| \\ &= \theta\|u_1 - u_2\|, \end{aligned} \tag{19}$$

where

$$\theta = \rho\lambda + \sqrt{1 - 2\rho\alpha + \beta^2\rho^2}.$$

From (9), it follows that $\theta < 1$, so the mapping w is a contraction mapping and consequently, it has a fixed point $w(u) = u \in H$ satisfying the problem (1).

□

Remark 3.1. *We point out that the solution of the auxiliary problem* (13) *is equivalent to finding the minimum of the functional* $I[w]$, *where*

$$I[w] = \frac{1}{2}\langle w - u, w - u\rangle - \rho\langle Lu - |u| - f, w - u\rangle,$$

which is a differentiable convex functional associated with the inequality (13). *This alternative formulation can be used to suggest iterative methods for solving the general absolute value equations. This auxiliary functional can be used to find a kind of gap function, whose stationary points solves the problem* (1), *see* [3].

It is clear that, if $w = u$, then w is a solution of (1). This observation shows that the auxiliary principle technique can be used to suggest the following iterative method for solving the generalized absolute value equations (1).

Algorithm 3.1. *For a given initial value* u_0, *compute the approximate solution* x_{n+1} *by the iterative scheme*

$$\langle Lu_n - \lambda|u_n| + u_{n+1} - u_n, v - u_{n+1}\rangle = \langle f, v - u_{n+1}\rangle, \forall v \in H.$$

From Algorithm 3.1, one can easily obtain the Picard type iterative method for solving the absolute value equation (4) and appears to be a new one.

Algorithm 3.2. *For a given initial value* u_0, *compute the approximate solution* x_{n+1} *by the iterative scheme*

$$u_{n+1} = u_n - \rho(Lu_n - \lambda|u_n| - f), \quad n = 0, 1, 2, 3...$$

We again use the auxiliary principle technique to suggest an implicit method for solving the problem (1). For a given $u \in H$ satisfying (1), consider the problem of finding $w \in H$ such that,

$$\langle \rho(Lw - \lambda|w|), v - w\rangle + \langle w - u, v - w\rangle = \rho\langle f, v - w\rangle, \quad \forall v \in H, \tag{20}$$

which is called the auxiliary problem. We note that the auxiliary problems (13) and (20) are quite different.

Clearly $w = u \in H$ is a solution of (1). This observation allows us to suggest the following iterative method for solving the problem (1).

Algorithm 3.3. *For a given initial value* u_0, *compute the approximate solution* x_{n+1} *by the iterative scheme*

$$\langle \rho Lu_{n+1} - \lambda\rho|u_{n+1}| + u_{n+1} - u_n, v - u_{n+1}\rangle = \langle \rho f, v - u_{n+1}\rangle, \forall v \in H, \tag{21}$$

which is an implicit method.

From this implicit method, we can obtain the following iterative method for solving (2)

Algorithm 3.4. *For a given initial value* u_0, *compute the approximate solution* x_{n+1} *by the iterative scheme*

$$u_{n+1} = u_n - \rho(Lu_{n+1} - \lambda|u_{n+1}| - f), \quad n = 0, 1, 2, 3...$$

This is a new implicit method for solving the absolute value equations (2).

To implement the implicit method, one uses the explicit method as a predictor and implicit method as a predictor. Consequently, we obtain the two-step method for solving the problem (1).

Algorithm 3.5. *For a given initial value* u_0, *compute the approximate solution* x_{n+1} *by the iterative schemes*

$$\begin{aligned}\langle \rho L u_n - \lambda\rho|u_n| + y_n - u_n, v - u_{n+1}\rangle &= \langle \rho f, v - y_n\rangle, \forall v \in H,\\ \langle \rho L y_n - \lambda\rho|y_n| + u_{n+1} - u_n, v - u_{n+1}\rangle &= \langle \rho f, v - u_{n+1}\rangle, \forall v \in H,\end{aligned}$$

which is known as two-step iterative method for solving problem (1).

Based on the above arguments, we can suggest a new two-step(predictor-corrector) method for solving the absolute value equations (2).

Algorithm 3.6. *For a given initial value* u_0, *compute the approximate solution* x_{n+1} *by the iterative schemes*

$$\begin{aligned}y_n &= u_n - \rho(Lu_n - \lambda|u_n| - f)\\ u_{n+1} &= u_n - \rho(Ly_n - \lambda|y_n| - f), \quad n = 0, 1, 2...\end{aligned}$$

We now consider the convergence analysis of Algorithm 3.3 and this is the main motivation of our next result.

Theorem 3.2. *Let* $u \in H$ *be a solution of problem* (1) *and let* u_{n+1} *be the approximate solution obtained from Algorithm 3.3. If* L *is a monotone operator, then*

$$(1 - 2\lambda\rho)\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2. \tag{22}$$

Proof. Let $u \in H$ be a solution of (1). Then

$$\langle Lu - \lambda|u|, v - u\rangle = \langle f, v - u\rangle, \quad \forall v \in H,$$

which implies that

$$\langle Lv - \lambda|u|, v - u\rangle \geq \langle f, v - u\rangle, \quad \forall v \in H, \tag{23}$$

since the operator L is monotone.
Taking $v = u_{n+1}$ in (23) and $v = u$ in (21), we have

$$\langle Lu_{n+1} - \lambda|u|, u_{n+1} - u\rangle \geq \langle f, u_{n+1} - u\rangle, \quad \forall v \in H, \tag{24}$$

and

$$\langle \rho L u_{n+1} - \rho\lambda|u_{n+1}| + u_{n+1} - u_n, u - u_{n+1}\rangle = \langle \rho f, u - u_{n+1}\rangle, \forall v \in H, \tag{25}$$

From (24) and (25), we have

$$\langle u_{n+1} - u_n, u - u_{n+1}\rangle \geq -\lambda\rho\langle |u_{n+1}| - |u|, u_{n+1} - u\rangle. \tag{26}$$

Using the relation $2ab = \|a + b\|^2 - \|a\|^2 - \|b\|^2, \forall a, b \in H$, the Cauchy-Swarchtz inequality and from (26), we have

$$(1 - 2\lambda\rho)\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2,$$

which is the required (22).

□

Theorem 3.3. *Let* $\bar{u} \in H$ *be a solution of* (1) *and let* u_{n+1} *be the approximate solution obtained from Algorithm 3.3. If* L *is a monotone operator and* $2\lambda\rho < 1$, *then*

$$\lim_{n\to\infty} u_{n+1} = \bar{u}. \tag{27}$$

Proof. Let $\bar{u} \in H$ be a solution of (1). From ((22), it follows that the sequence $\{\|\bar{u} - u_n\|\}$ is noncreasing and consequently the sequence $\{u_n\}$ is bounded. Also, from (22), we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leq \|u_0 - \bar{u}\|^2,$$

which implies that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{28}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequences $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converges to $\bar{u} \in H$. Replacing u_n by u_{n_j}in (21), taking the limit as $n_j \to \infty$ and using (28), we have

$$\langle L\hat{u} - \lambda|\hat{u}|, v - \bar{u}\rangle = \langle f, v - \hat{u}\rangle, \quad \forall v \in H,$$

which shows that $\hat{u} \in H$ satisfies (1) and

$$\|u_{n+1} - u_n\|^2 \leq \|u_n - \hat{u}\|^2.$$

From the above inequality, it follows that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n\to\infty} u_n = \hat{u}$. □

Conclusion

In this paper, we have considered a new class of absolute value equations. The auxiliary principle technique has been used to study the existence of the unique solution of the generalized absolute value equations. Some new iterative methods are suggested for solving the absolute value equations. The convergence analysis of these iterative methods is investigated under suitable conditions. This is a new approach for the Lax-Milgram lemma. We would like to emphasize that the results obtained and discussed in this paper may motivate a number of novel applications and extensions and in these areas.

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