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RESEARCH ON A CLASS OF THREE-POINT BOUNDARY VALUE PROBLEM VIA FIXED POINT INDEX THEOREM

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Many phenomenons in economics, astronomy, biology, engineering, mechanics and other fields can be described by the multi-point boundary value problems (BVPs). It is difficult to obtain the analytical solution of the multi-point BVPs, so the discussion of the existence and uniqueness of solution will become the fundamental issue of BVPs. Since Gupta [1] began to study second-order three-point BVP for nonlinear differential equations, more and more scholars conduct a systematic study of three-point BVPs, and achieve fruitful results. In reference [12], Sun studied a singular second-order three-point BVP, and gets some existence results of positive solutions. In this paper, motivated by Sun's results, we consider a second-Order three-Point BVP with sign-changing nonlinear terms. We get some existence results of symmetric positive solutions by using the theory of fixed point index, and take a specific case for example to illustrate the application of these results. The results generalize and improve Sun's results.

Keywords: Three-point boundary value problem; Sign-changing nonlinear terms; Symmetric positive solution; Fixed point index

1. Introduction

Along with society's progress and the development of science and technology, many phenomenons in economics, astronomy, biology, engineering, mechanics and other fields can be studied by establishing mathematical models of the boundary value problems (BVPs). The existence and uniqueness of the solution of BVPs will become hot issue, because the analytical solution of BVPs is difficult to obtain.

Since Gupta [1] began to study second-order three-point BVP for nonlinear differential equations, more and more scholars conduct a systematic study of three-point BVPs, and achieve fruitful results (see, for example, [2-11]). In the process of solving the practical problems, the existence of solutions for BVPs is an important question, especially the existence of positive solutions. Much literature has a detailed introduction of these problems, such as [12-19].

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Sun [12] obtained some existence results of positive solutions for a second-order three-point BVP. In this paper, we consider a more complicated second-order three-point BVP:

$$v''(\tau)+\alpha(\tau)g\left(v(\tau)\right)+\beta(\tau)\mu(\tau)=0,\ \ 0<\tau<1, \tag{1.1}$$

$$v(0)-v(1)=0\,,\, v'(0)-v'(1)=v\left(\frac{1}{2}\right), \tag{1.2}$$

Where, $\alpha(\tau)$ and $\mu(\tau)$ are symmetric on $(0,1)$ and may be singular at $\tau=0$ and $\tau=1$, $\beta(\tau)$ is symmetric and continuous on $[0,1]$, the range of $\alpha(\tau)$, $\beta(\tau)$, $\mu(\tau)$ is $[0,+\infty)$.

$$g:[0,+\infty)\to[0,+\infty),$$

is continuous.

2. Common definitions and lemmas

For convenience, we recall below some definitions which are commonly used in this research field.

Definition 2.1 [12] $\mathrm{M}\subset\Xi$ is a nonempty closed convex, where Ξ is a real Banach space, if M satisfies the following two conditions:

(1) $y\in\mathrm{M}$, $\lambda>0$, implies $\lambda y\in\mathrm{M}$,

(2) $y\in\mathrm{M}$, $-y\in\mathrm{M}$, implies $y=0$,

then, we call M a cone.

Definition 2.2[12] An operator Λ is called completely continuous, if it is continuous, and maps bounded sets into precompact sets.

Definition 2.3[12] A function f is said to be symmetric on $[0,1]$, if it satisfies the following condition:

$$f(\tau)=f(1-\tau),\ \tau\in[0,1].$$

Definition 2.4[12] A function f is said to be concave on $[0,1]$, if it satisfies the following condition:

$$f\left(\theta\tau_1+(1-\theta)\tau_2\right)\ge\theta f(\tau_1)+(1-\theta)f(\tau_2),\ \theta,\tau_1,\tau_2\in[0,1].$$

Definition 2.5 Denote

$$C^+[0,1]=\left\{f\in C[0,1]\,\middle\|\, f(\tau)\ge 0,\tau\in[0,1]\right\},$$

in which the Banach space $Z=C[0,1]$, equipped with $\|v\|=\max\limits_{\tau\in[0,1]}|v(\tau)|$.

The following results introduced by Sun [12] will be used to prove our main results.

Lemma 2.1[12] Let $z \in C[0,1]$. Then, the three-point BVP

$$-v''(\tau) = z(\tau),\ \ 0 < \tau < 1, \tag{2.1}$$

$$v(0) - v(1) = 0\ , v'(0) - v'(1) = v\left(\frac{1}{2}\right), \tag{2.2}$$

has a unique solution

$$v(\tau) = \int_0^1 H(\tau,\xi) z(\xi) d\xi\ , \tag{2.3}$$

where,

$$H(\tau,\xi) = H_1(\tau,\xi) + H_2(\xi),$$

$$H_1(\tau,\xi) = \begin{cases} \xi(1-\tau), & 0 \le \xi \le \tau \le 1, \\ \tau(1-\xi), & 0 \le \tau \le \xi \le 1, \end{cases}$$

$$H_2(\xi) = \begin{cases} 1 - \dfrac{\xi}{2}, & 0 \le \xi \le \dfrac{1}{2}, \\ \dfrac{1+\xi}{2}, & \dfrac{1}{2} \le \xi \le 1. \end{cases}$$

Lemma 2.2[12] Let $\xi, \tau \in [0,1]$, then

(1) $H(\tau,\xi) \ge \dfrac{3}{4}$;

(2) $\dfrac{3}{4} H(\xi,\xi) \le H(\tau,\xi) \le H(\xi,\xi)$;

(3) $H(1-\tau, 1-\xi) = H(\tau,\xi)$.

Lemma 2.3[12] If the BVP (2.1), (2.2) has a unique solution $v(\tau)$, and if the function $z(\tau)$ in (2.1) satisfies $z(\tau) \in C^+[0,1]$. Then, $v(\tau)$ satisfies

$$\min_{\tau \in [0,1]} v(\tau) \ge \frac{3}{4}\|v\|.$$

Lemma 2.4[12] If $z(\tau)$ in equation (2.1) satisfies

$$z(\tau) = z(1-\tau),\ \tau \in [0,1],$$

and, if the BVP (2.1), (2.2) has a unique solution $v(\tau)$, then, $v(\tau)$ satisfies

$$v(\tau)=v(1-\tau),\ \tau\in[0,1],$$

i.e., $v(\tau)$ is symmetric on $[0,1]$.

Lemma 2.5[2] Let Ξ be a real Banach space, Z be a retract of Ξ and V be a bounded relatively open subset of Z. Assuming that

$$B:\bar{V}\to Z,$$

is a completely continuous operator, and has no fixed points on ∂V. Then, there exists an integer $i(B,V,Z)$ satisfying the following conditions:

(1) If $\forall y\in\bar{V}$, $By\equiv z_0\in V$, then, $i(B,V,Z)=1$;

(2) If X is a retract of Z and $B(\bar{V})\in X$, then, $i(B,V,Z)=i(B,V\cap X,X)$;

(3) If $G:[0,1]\times\bar{V}\to Z$ is completely continuous and $(\tau,y)\in[0,1]\times\partial V$, $G(\tau,y)\neq y$, $i(G(\tau,\cdot),V,Z)$ is independent of $\tau(0\le\tau\le1)$;

(4) If V_1 and V_2 are disjoint open subsets of V such that B has no fixed points on $\bar{V}\setminus(V_1\cup V_2)$, then, $i(B,V,Z)=i(B,V_1,Z)+i(B,V_2,Z)$;

(5) If V_0 is an open subset of V such that B has no fixed points in $\bar{V}\setminus V_0$, then, $i(B,V,Z)=i(B,V_0,Z)$;

(6) B has at least one fixed point in V whenever $i(B,V,Z)\neq0$.

Lemma 2.5 establishes the existence and uniqueness of the fixed point index.

3. Main results

In this section, for research purposes, we make the following assumptions:

(A_1) $\alpha:(0,1)\to[0,+\infty)$ is continuous, symmetric on (0,1) and

$$\int_0^1 H(\xi,\xi)\beta(\xi)\mu_-(\xi)d\xi=\theta_0,$$

$$\int_0^1 H(\xi,\xi)\left[\alpha(\xi)g(1)+\beta(\xi)\mu_+(\xi)\right]d\xi\le\frac{\theta}{(\theta+1)^{\lambda_1}+1}. \tag{3.1}$$

Where, $\mu_+(\tau)=\max\{\mu(\tau),0\}$, $\mu_-(\tau)=\max\{-\mu(\tau),0\}$.

(A_2) $g(1)>0$, there exist constants $\lambda_1\ge\lambda_2>1$, such that, $\forall v\in[0,+\infty)$,

$$c^{\lambda_1}g(v)\le g(cv)\le c^{\lambda_2}g(v),\ \forall 0\le c<1.$$

The proof of the following fixed point results can be found in Ref. [2, pp. 88–89]

Lemma 3.1[2] Let Ω (with $0\in\Omega$) be a bounded open subset of Z, Q a cone in Z, and $B:\bar{\Omega}\bigcap Q\to Q$ a completely continuous operator.

(1) $\forall v\in\partial\Omega\bigcap Q$ and $\lambda\geq 1$, if $Bv\neq\lambda v$,

then,

$$i\left(B,\Omega\bigcap Q,Q\right)=1.$$

(2) $\forall v\in\partial\Omega\bigcap Q$, $\forall\lambda\in\left(0,1\right]$, if $Bv\neq\lambda v$, and $\inf\limits_{v\in\partial\Omega\bigcap Q}\left\|Bv\right\|>0$

then,

$$i\left(B,\Omega\bigcap Q,Q\right)=0.$$

Let

$$Q=\left\{y\in Z: y\left(\tau\right)\text{ is symmetric, }\min_{\tau\in[0,1]}y\left(\tau\right)\ \geq\frac{3}{4}\left\|y\right\|\right\}.$$

Clearly, Q is a cone of Z.

Set

$$f\left(\tau\right)=\int_0^1 H\left(\tau,\xi\right)\beta\left(\xi\right)\mu_-\left(\xi\right)d\xi,$$

then, $f\left(\tau\right)$ is symmetric on [0,1]. And, for $z\left(\tau\right)=\alpha\left(\tau\right)\mu_-\left(\tau\right)$, $f\left(\tau\right)$ is the unique solution of BVP (2.1), (2.2).

Now, we consider the singular second-order three-point BVP

$$-v''\left(\tau\right)=\alpha(\tau)g\left(\left[v\left(\tau\right)-f\left(\tau\right)\right]^*\right)+\beta(\tau)\mu_+\left(\tau\right),\ 0<\tau<1, \tag{3.2}$$

$$v\left(0\right)-v\left(1\right)=0,\ v'\left(0\right)-v'\left(1\right)=v\left(\frac{1}{2}\right), \tag{3.3}$$

where,

$$\left[v\left(\tau\right)-f\left(\tau\right)\right]^*=\max\left\{v\left(\tau\right)-f\left(\tau\right),0\right\}.$$

Lemma 3.2 Let $f\left(\tau\right)=\int_0^1 H\left(\tau,\xi\right)\beta\left(\xi\right)\mu_-\left(\xi\right)d\xi$. If $\forall\tau\in\left[0,1\right], v\left(\tau\right)\geq f\left(\tau\right)$, and $v\left(\tau\right)$ is a positive and symmetric solution of the BVP (3.2), (3.3).Then,

$$z\left(\tau\right)=v\left(\tau\right)-f\left(\tau\right)$$

is a solution of the BVP (1.1), (1.2).

Proof $\forall \tau \in [0,1]$, it's easy to get $z(\tau) = v(\tau) - f(\tau) \geq 0$, on account of $v(\tau) \geq f(\tau)$.

Note that $f(\tau)$ is symmetric on [0,1], and according to Lemma 2.1, for $z(\tau) = \alpha(\tau)\mu_-(\tau)$, $f(\tau)$ is the unique solution of BVP (2.1), (2.2), we get

$$-\left(z(\tau) + f(\tau)\right)'' = \alpha(\tau) g\left(z(\tau)\right) + \beta(\tau)\mu_+(\tau),$$

namely,

$$-z''(\tau) - f''(\tau) = \alpha(\tau) g\left(z(\tau)\right) + \beta(\tau)\mu_+(\tau),$$

which implies that

$$-z''(\tau) + \beta(\tau)\mu_-(\tau) = \alpha(\tau) g\left(z(\tau)\right) + \beta(\tau)\mu_+(\tau).$$

So,

$$-z''(\tau) - f''(\tau) = \alpha(\tau) g\left(z(\tau)\right) + \beta(\tau)\mu_+(\tau).$$

That is to say, $z(\tau) = v(\tau) - f(\tau)$ is a solution of the singular second-order three-point BVP (1.1), (1.2). □

Remark 3.1 As in Ref. [13], we point out that if $g(v)$ satisfies (A_2), then $\forall v_1 \leq v_2$ in $[0, +\infty)$, $g(v_1) \leq g(v_2)$, and $\lim_{v \to +\infty} \frac{g(v)}{v} = +\infty$.

Remark 3.2 Since for all $v \in Z$, choose $0 < \rho < 1$ such that $\rho \|v\| < 1$, then,

$$\rho\left[v(\tau) - f(\tau)\right]^* \leq \theta v(\tau) < 1.$$

So, by (A_2) and Remark 3.1, we have

$$g\left(\left[v(\tau) - f(\tau)\right]^*\right) \leq \left(\frac{1}{\rho}\right)^{\lambda_1} g\left(\rho\left[v(\tau) - f(\tau)\right]^*\right) \leq \rho^{\lambda_2 - \lambda_1} \|v\|^{\lambda_2} g(1).$$

Lemma 3.3 Let (A_1) and (A_2) hold true, then, the operator $\Phi : Z \to Z$ which defined by

$$\Phi v(\tau) = \int_0^1 H(\tau, \xi)\left[\alpha(\xi) g\left(v(\xi)\right) + \beta(\xi)\mu_+(\xi)\right] d\xi \tag{3.4}$$

is a completely continuous operator. Furthermore, $\Phi(Q) \subset Q$.

Proof Firstly, by taking the second-order derivatives with respect to τ on both sides of equation (3.4), we get

$$(\Phi v)''(\tau) = -\alpha(\tau) g\left(v(\tau)\right) - \beta(\tau)\mu_+(\tau) \le 0,$$

which implies that Φv is concave. Applying the result of Lemma 2.3, we have

$$\min_{\tau\in[0,1]} \Phi v(\tau) \ge \frac{3}{4}\|\Phi v\|, \text{ for all } \tau \in [0,1],$$

which implies $\Phi(Q) \subset Q$.

Secondly, we claim that Φ is completely continuous. To prove that, $\forall n \ge 2$, define $\alpha_n(\tau)$ and $\mu_{n+}(\tau)$ as following:

$$\alpha_n(\tau) = \begin{cases} \inf_{0\le\xi\le\frac{1}{n}} \alpha(\xi), & 0 \le \tau \le \frac{1}{n}, \\ \alpha(\xi), & \frac{1}{n} \le \tau \le 1 - \frac{1}{n}, \\ \inf_{1-\frac{1}{n}\le\xi\le 1} \alpha(\xi), & 1 - \frac{1}{n} \le \tau \le 1, \end{cases}$$

$$\mu_{n+}(\tau) = \begin{cases} \inf_{0\le\xi\le\frac{1}{n}} \mu_+(\xi), & 0 \le \tau \le \frac{1}{n}, \\ \mu_+(\xi), & \frac{1}{n} \le \tau \le 1 - \frac{1}{n}, \\ \inf_{1-\frac{1}{n}\le\xi\le 1} \mu_+(\xi), & 1 - \frac{1}{n} \le \tau \le 1, \end{cases}$$

and $\Phi_n : Q \to Q$ by

$$\Phi_n v(\tau) = \int_0^1 H(\tau,\xi)\left[\alpha_n(\xi) g\left(v(\xi)\right) + \beta(\xi)\mu_{n+}(\xi)\right] d\xi.$$

Then, Φ_n is compact on Q for all $n \ge 2$, by applying the Ascoli-Arzela Theorem.

Next, we're going to prove that Φ_n converges uniformly to Φ as $n \to +\infty$. In fact, because of

$$\int_0^1 H(\xi,\xi)\left[\alpha(\xi) g(1) + \beta(\xi)\mu_+(\xi)\right] d\xi < +\infty,$$

we get

$$\int_0^1 \mathrm{H}(\xi,\xi)\alpha(\xi)g(1)d\xi < +\infty,$$

and

$$\int_0^1 H(\xi,\xi)\beta(\xi)\mu_+(\xi)d\xi < +\infty.$$

Set

$$e\left(\frac{1}{n}\right) = \left[0,\frac{1}{n}\right] \bigcup \left[1-\frac{1}{n},1\right],$$

by the absolute continuity of integral, we have

$$\lim_{n\to+\infty}\int_{e\left(\frac{1}{n}\right)} \alpha(\xi)H(\xi,\xi)d\xi = 0,$$

and

$$\lim_{n\to+\infty}\int_{e\left(\frac{1}{n}\right)} \beta(\xi)H(\xi,\xi)\mu_+(\xi)d\xi = 0.$$

Denote $D_\Theta = \{v\in Q : \|v\| \le \Theta\}$, $O_\Theta = \max\{g(y): 0 \le y \le \Theta\} < +\infty$.

Notice the fact that $H(\tau,\xi) \le H(\xi,\xi)$ ($\forall \tau,\xi \in [0,1]$), so, $\forall \tau \in [0,1]$, fixed $\Theta > 0$ and $v \in D_\Theta$, and let (A_1) and (A_2) hold true. Notice the fact that $H(\tau,\xi) \le H(\xi,\xi)$ for all $\tau,\xi \in [0,1]$, we have

$$\begin{aligned}
&\left|\Phi_n v(\tau) - \Phi v(\tau)\right| \\
&= \left|\int_0^1 [\alpha(\xi) - \alpha_n(\xi)]H(\tau,\xi)g(v(\xi))d\xi + \int_0^1 [\mu_+(\xi) - \mu_{n+}(\xi)]H(\tau,\xi)\beta(\xi)d\xi\right| \\
&\le \left|\int_0^1 [\alpha(\xi) - \alpha_n(\xi)]H(\tau,\xi)g(v(\xi))d\xi\right| + \left|\int_0^1 [\mu_+(\xi) - \mu_{n+}(\xi)]G(\tau,\xi)\beta(\xi)d\xi\right| \\
&\le O_\Theta \int_{e\left(\frac{1}{n}\right)} \alpha(\xi)H(\xi,\xi)d\xi + \int_{e\left(\frac{1}{n}\right)} \beta(\xi)H(\xi,\xi)\mu_+(\xi)d\xi \\
&\quad \to 0,\ n \to +\infty.
\end{aligned}$$

That is to say, Φ_n converges uniformly to Φ as $n \to +\infty$ on any bounded subset of Q. Since Φ_n are completely continuous operators. So, Φ is completely continuous. □

Lemma 3.4 Let

$$\Omega_\theta = \{ y \in Q : \|y\| \le \theta \},$$

where $\theta = \frac{4}{3}\theta_0$. Then, $i(\Phi, \Omega_\theta, Q) = 1$.

Proof $\forall v \in \Omega_\theta$, we have $\|v\| \le \theta$. Since $\forall \theta \in [0,1]$,

$$[v(\tau) - f(\tau)]^* \le v(\tau) \le \|v\| \le \theta < \theta + 1,$$

by Remark 3.2, we have

$$\begin{aligned}
&\alpha(\tau) g([v(\tau) - f(\tau)]^*) + \beta(\tau)\mu_+(\tau) \\
&\le \alpha(\tau) g(\theta+1) + \beta(\tau)\mu_+(\tau) \\
&\le (\theta+1)^{\lambda_1} \alpha(\tau) g(1) + \beta(\tau)\mu_+(\tau) \\
&\le \left[(\theta+1)^{\lambda_1} + 1\right]\left[\alpha(\tau) g(1) + \beta(\tau)\mu_+(\tau)\right].
\end{aligned}$$

Now, we will show that

$$\mu x \ne \Phi x,\ \mu \ge 1,\ x \in \partial\Omega_\theta.$$

In fact, if $\mu x = \Phi x$, then, there exists $\mu \ge 1$, $x_0 \in \partial\Omega_\theta$, such that $\mu x_0 = \Phi x_0$, i.e., $x_0 = \frac{1}{\mu}\Phi x_0$ and $0 \le \frac{1}{\mu} \le 1$. Thus, we have

$$\begin{aligned}
\theta &= \|x_0\| \\
&\le \sup_{\theta\in[0,1]} \left| \int_0^1 H(\tau,\xi)\left[\alpha(\xi) g([x_0(\xi) - f(\xi)]^*) + \beta(\xi)\mu_+(\xi)\right] d\xi \right| \\
&\le \left[(\theta+1)^{\lambda_1} + 1\right] \int_0^1 H(\xi,\xi)\left[\alpha(\xi) g(1) + \beta(\xi)\mu_+(\xi)\right] d\xi .
\end{aligned}$$

So,

$$\int_0^1 H(\xi,\xi)\left[\alpha(\xi) g(1) + \beta(\xi)\mu_+(\xi)\right] d\xi \ge \frac{\theta}{(\theta+1)^{\lambda_1} + 1}. \tag{3.5}$$

The above inequality is a contradiction of (3.1). Using Lemma 3.1, we see that, $i(\Phi, \Omega_\theta, Q) = 1$. □

Lemma 3.5 There exists a constant $\Theta > \theta$, such that $i\left(\Phi, \Omega_{\Theta}, Q\right) = 0$, where

$$\Omega_{\Theta} = \left\{ y \in Q : \|y\| \leq \Theta \right\}.$$

Proof Choose a constant N such that,

$$N > \frac{32}{15}\left[\min_{\tau \in [0,1]} \int_0^1 H\left(\tau, \xi\right) \alpha\left(\xi\right) d\xi\right]^{-1}.$$

By Remark 3.1, there exists $\Theta_1 > 2\theta$, when $y \geq \Theta_1$, such that

$$\frac{g\left(v\right)}{v} \geq N .$$

That is

$$g\left(v\right) \geq Nv ,\ y \geq \Theta_1 .$$

Let $\Theta > \Theta_1$, obviously, $\Theta > \Theta_1 > 2\theta$. Thus $\frac{\theta}{\Theta} < \frac{1}{2}$, so $\frac{\theta_0}{\Theta} < \frac{3}{8}$.

Now, we show that $v < \Phi v$, $v \in \partial\Omega_{\Theta}$. Otherwise, $\exists z_1 \in \partial\Omega_{\Theta}$, such that $z_1 \geq \Phi z_1$.

Using Lemma 2.2, $\forall \tau \in [0,1]$, we have

$$\begin{aligned} z_1\left(\tau\right) - f\left(\tau\right) &\geq z_1\left(\tau\right) - \int_0^1 H\left(\xi, \xi\right) \beta\left(\xi\right) \mu_-\left(\xi\right) d\xi \\ &\geq z_1\left(\tau\right) - \theta_0 \geq z_1\left(\tau\right) - \frac{z_1\left(\tau\right)}{\|z_1\|}\theta_0 \\ &\geq \left(1 - \frac{\theta_0}{\Theta}\right) z_1\left(\tau\right) \geq \frac{1}{8} z_1\left(\tau\right) \geq \frac{15}{32}\|z_1\| \\ &= \frac{15}{32}\Theta > 0. \end{aligned}$$

So,

$$\begin{aligned} \Theta &\geq z_1\left(\tau\right) \geq \Phi z_1\left(\tau\right) \\ &= \int_0^1 H\left(\tau, \xi\right)\left[\alpha(\xi) g\left(\left[z_1\left(\xi\right) - f\left(\xi\right)\right]^*\right) + \beta(\xi)\mu_+\left(\xi\right)\right] d\xi \\ &= \int_0^1 H\left(\tau, \xi\right)\left[\alpha(\xi) g\left(z_1\left(\xi\right) - f\left(\xi\right)\right) + \beta(\xi)\mu_+\left(\xi\right)\right] d\xi \\ &\geq \int_0^1 H\left(\tau, \xi\right) \alpha(\xi) g\left(z_1\left(\xi\right) - f\left(\xi\right)\right) d\xi \end{aligned}$$

$$\geq \int_0^1 H(\tau,\xi)\alpha(\xi)N\left(z_1(\xi)-f(\xi)\right)d\xi$$
$$\geq \int_0^1 H(\tau,\xi)\alpha(\xi)N\frac{15}{32}\Theta d\xi,\ \tau\in[0,1].$$

Consequently

$$\Theta \geq \frac{15}{32}\Theta N \min_{\tau\in[0,1]}\int_0^1 H(\tau,\xi)\alpha(\xi)d\xi.$$

That is

$$N \leq \frac{32}{15}\left[\min_{\theta\in[0,1]}\int_0^1 H(\tau,\xi)\alpha(\xi)d\xi\right]^{-1}.$$

This contradicts N that we choose. Using Lemma 3.1, we get $i\left(\Phi,\Omega_\Theta,Q\right)=0$. □

Theorem 3.1 Assume that the conditions (A_1) and (A_2) hold. Then, the second-order three-point BVP (1.1), (1.2):

$$v''(\tau)+\alpha(\tau)g\left(v(\tau)\right)+\beta(\tau)\mu(\tau)=0,\quad 0<\tau<1,$$
$$v(0)-v(1)=0,\ v'(0)-v'(1)=v\left(\frac{1}{2}\right),$$

has at least one symmetric positive solution.

Proof. Take notice of the definition of the fixed point index, and using Lemma 3.4 -3.5, we see that

$$i\left(\Phi,\Omega_\Theta\setminus\bar{\Omega}_\theta,Q\right)=-1.$$

Thus, Φ has a fixed point x_0 in $\Omega_\Theta\setminus\bar{\Omega}_\theta$, with $\theta<\|x_0\|<\Theta$.

Since $\theta<\|x_0\|$, we have

$$x_0(\tau)-f(\tau)\geq\frac{3}{4}\|x_0\|-\int_0^1 H(\tau,\xi)\beta(\xi)\mu_-(\xi)d\xi$$
$$>\frac{3}{4}\times\frac{4}{3}\theta-\theta=0,\ \tau\in[0,1].$$

Let $v(\tau)=x_0(\tau)-f(\tau)$. Using Lemma 3.4, we see that $v(\tau)$ is a symmetric positive solution of the singular second-order three-point BVP (1.1), (1.2). □

In the following, we will consider an example to illustrate the application of theorem 3.1.

Example 3.1 Consider a second-order three-point BVP as following:

$$-v(\tau)=\frac{24}{775}\min\{\tau,1-\tau\}v^2+\min\{\tau,1-\tau\}\left(-\frac{1}{\tau(1-\tau)}\right),\ 0<\tau<1, \tag{3.6}$$

$$v(0)-v(1)=0\,,v'(0)-v'(1)=v\left(\frac{1}{2}\right). \tag{3.7}$$

We will prove the above BVP has at least one symmetric and positive solution.

Proof In fact, set

$$\alpha(\tau)=\frac{24}{775}\min\{\tau,1-\tau\},\ \beta(\tau)=\min\{\tau,1-\tau\},$$

and

$$\mu(\tau)=-\frac{1}{\tau(1-\tau)}.$$

A direct computation shows that

$$\theta_0=\frac{25}{24}+\ln 4,\ g(1)=1,\ \theta=\frac{4}{3}\theta_0=\frac{25}{18}+\frac{4}{3}\ln 4\,,$$

$$\int_0^1 H(\xi,\xi)\left[\alpha(\xi)g(1)+\beta(\xi)\mu_+(\xi)\right]d\xi=\frac{31}{24}.$$

Set $\lambda_1=2$, $\lambda_2<2$, therefore, the condition (A_1), (A_2) hold. □

4. Conclusions

Firstly, this paper introduces the application background of multi-point BVPs, and the recent research achievements of three-point BVPs.

Then, we introduce a second-Order three-Point BVP with sign-changing nonlinear terms, which improved the BVP in Sun [12]. In order to prove Theorem 3.1, we review several lemmas appearing in Sun [12], and prove a series of lemma by using the fixed point theorem.

Finally, by Theorem 3.1, we establish the existence of symmetric positive solutions of the BVP (1.1), (1.2), and take a specific case for example to illustrate the application of Theorem 3.1.

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