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INJECTIVITY OF BEURLING AND WEIGHTED MEASURE ALGEBRAS

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For a locally compact group G let $L^1(G,\omega)$ be a Beurling algebra. We characterize injectivity property of $L^1(G,\omega)$, $M(G,\omega)$ and $L^1(G,\omega)''$ as a Banach $L^1(G,\omega)$-Modules. This characterization is employed to find a necessary and sufficient condition for amenability of G. In the special case where $\{\omega_n\}_{n=1}^{\infty}$ is a sequence of weight functions on G we prove the same result for Fréchet algebras $A(\omega)$ and $B(\omega)$.

Keywords: Injective module, Beurling algebra, weighted measure algebra, amenable group.

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1. Introduction and Preliminaries

Homology properties of some class of Banach algebras such as $L^1(G)$, $L^p(G)(1 < p < \infty)$, $M(G)$ and $L^1(G)''$ has been studied and expounded by H. G. Dales and M. E. Polyakov [4], and others [2], undertook a study of these properties for locally compact group G. The development of the homology theory of topological algebras outside the framework of Banach structures was introduced by Taylor [11] and then appeared in the works of Helemskii [6]. Unlike of the category of Banach algebras there are some problem on homology properties of locally convex modules over non-normable algebras, for example given a Fréchet module X over a Fréchet algebra $\mathcal{A}$, the dual module X', has no reasonable topology to make it a Fréchet space. Moreover, the action of $\mathcal{A}$ on X' often fails to be jointly continuous with respect to any natural topology on X' (cf. [11]). Let us also remark that many Fréchet algebras do not have nontrivial injective Fréchet modules at all [8].

In this work we investigate the relationship between injectivity of some Banach left $L^1(G,\omega)$-module and locally compact group G related to the work of Dales and Polyakov[4]. In fact for a locally compact group G and a weight function ω on it with condition $\omega \geq 1$, we show that $M(G,\omega)$ is injective as a $L^1(G,\omega)$-module if and only if G is amenable and ω is bounded. A similar

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result will be proved for $L^1(G,\omega)$ and $L^1(G,\omega)''$. Finally we look into the injectivity of Fréchet algebras $A(\omega)=\cap_1^\infty L^1(G,\omega_n)$ and $B(\omega)=\cap_1^\infty M(G,\omega_n)$ where $\{\omega_n\}$ is an increasing sequence of weight functions on G. We begin by recalling some basic terminology.

Let G be a locally compact group and ω be a weight function on G, that is a positive continuous function with $\omega(xy)\leq\omega(x)\omega(y)$ for all $x,y\in G$ and $\omega(e_G)=1$ where e_G is the identity of G. Throughout this paper ω is assumed to be $\omega(x)\geq 1$ and $\tilde{\omega}(x)=\omega(x^{-1})$ for all $x\in G$. The spaces $L^\infty(G,\frac{1}{\omega})$ and $L^1(G,\omega)$ will be defined by the set of all Borel measurable functions on G such that $||f||_{1,\omega}=\int_G|f(x)|\omega(x)dm(x)<\infty$ and $||f||_{\infty,\omega}=ess\sup_{x\in G}\frac{|f(x)|}{\omega(x)}<\infty$ respectively where m is left Haar measure on G. We identify two elements in $L^\infty(G,\frac{1}{\omega})$ if they are equal locally almost everywhere and in $L^1(G,\omega)$ if they are equal almost everywhere with respect to the left Haar measure m on G. Then $(L^1(G,\omega),||\cdot||_\omega)$ with convolution product, $f\star g(x)=\int_G f(y)g(y^{-1}x)dm(y)$, and $(L^\infty(G,\frac{1}{\omega}),||\cdot||_{\infty,\omega})$ with pointwise product are Banach algebras. $L^\infty(G,\frac{1}{\omega})$ is the dual space of $L^1(G,\omega)$ by following dual pair $\langle f,g\rangle=\int_G f(x)g(x)dm(x)$, where $f\in L^1(G,\omega)$ and $g\in L^\infty(G,\frac{1}{\omega})$. The left and right module actions of $L^1(G,\omega)$ on $L^\infty(G,\frac{1}{\omega})$ are defined by

$$f\cdot g(x)=\int_G f(y)g(xy)dm(y),\qquad g\cdot f(x)=\int_G f(y)g(yx)dm(y),$$

where $f\in L^1(G,\omega)$ and $g\in L^\infty(G,\frac{1}{\omega})$. Since the dual of any Banach $L^1(G,\omega)$-module is Banach $L^1(G,\omega)$- module so the dual of $L^\infty(G,\frac{1}{\omega})$ is also $L^1(G,\omega)$-module.

We denote by $M(G,\omega)$ the Banach space of all complex-valued, regular Borel measures μ on G such that $||\mu||_\omega=\int_G\omega(x)d|\mu|(x)<\infty$. Moreover suppose that

$$C_0(G,\frac{1}{\omega})=\{f\in L^\infty(G,\frac{1}{\omega}):\frac{f}{\omega}\in C_0(G)\}.$$

Then $C_0(G,\frac{1}{\omega})$ is a closed subspace of $L^\infty(G,\frac{1}{\omega})$ that is a left Banach $L^1(G,\omega)$-module. $M(G,\omega)$ is the dual of $C_0(G,\frac{1}{\omega})$ with respect to the pairing

$$\langle f,\mu\rangle=\int_G f(x)d\mu(x),\qquad f\in C_0(G,\frac{1}{\omega}),\mu\in M(G).$$

The convolution product $\star$ on $M(G,\omega)$ is defined by the formula $\langle f,\mu\star\nu\rangle=\int_G\int_G f(xy)d\mu(x)d\nu(y)$, where $\mu,\nu\in M(G,\omega)$ and $f\in C_0(G,\frac{1}{\omega})$. It is easy to see that the space of discrete measures $\ell^1(G,\omega)$ is a closed subspace of $M(G,\omega)$. We consider $M(G,\omega)$ as a Banach $L^1(G,\omega)$-module by the module actions $(f\star\mu)(x)=\int_G f(xy^{-1})\Delta_G(y^{-1})d\mu(y)$, and $(\mu\star f)(x)=\int_G f(y^{-1}x)d\mu(y)$, where $x\in G$, $f\in L^p(G,\omega)$, $1\leq p\leq\infty$, $\mu\in M(G,\omega)$ and Δ_G is modular function of G. A net $\{\mu_\alpha\}$ in $M(G,\omega)$ is called convergent to μ in *so*-topology if for all $a \in L^1(G,\omega)$:

$$||a \cdot \mu_\alpha - a \cdot \mu||_{\omega_n} \longrightarrow 0$$

Theorem 1.1. [3, Theorem 7.9.] *Let ω be a weight function on a locally compact group G. Then the Banach space $M(G,\omega)$ is a unital Banach algebra with respect to the convolution product $\star$; $L^1(G,\omega)$ is a closed ideal in $M(G,\omega)$, and $\ell^1(G,\omega)$ is a closed subalgebra of $M(G,\omega)$.*

2. Injective Fréchet modules

A complete topological space X whose topology is given by an increasing countable family of semi-norms is called a Fréchet space. Suppose that $\mathcal{A}$ is a Banach algebra, a complete Hausdorff locally convex space X is called a left $\mathcal{A}$-module if it is an algebraic left module over $\mathcal{A}$ and if in addition the action $m : \mathcal{A}\times X \to X$ is jointly continuous. The space X', the dual space of a Fréchet $\mathcal{A}$-bimodule X with the strong topology, is a locally convex $\mathcal{A}$-bimodule with respect to the module operations defined by

$$\langle a \cdot f, x\rangle = \langle f, x \cdot a\rangle, \qquad \langle f \cdot a, x\rangle = \langle f, a \cdot x\rangle \qquad (x \in X),$$

where $a \in \mathcal{A}$ and $f \in X'$. In this case we say that X' is the dual module of X. In special case Banach space E is a Banach $\mathcal{A}$-module, if E is a Banach space for a norm $\|\cdot\|$ and $\|a \cdot x\| \leq \|a\| \, \|x\|$ and $\|x \cdot a\| \leq \|a\| \, \|x\|$ for all $a \in \mathcal{A}$ and $x \in E$. If X is a Banach $\mathcal{A}$-module, then the dual Banach space X' is a Banach $\mathcal{A}$-module. An especially interesting case occurs when we consider $\mathcal{A}'$ and $\mathcal{A}''$ as Banach $\mathcal{A}$-modules.

For Fréchet spaces X and Y we denote the space of all continuous morphisms from X into Y by $\mathfrak{B}(X,Y)$, which is complete locally convex space ([11], p.159). If for Banach algebra $\mathcal{A}$, X and Y are $\mathcal{A}$-modules then the space $\mathfrak{B}(X,Y)$ is also $\mathcal{A}$-module [11, Proposition 3.1] with the following actions

$$a \cdot T(x) = T(x \cdot a), \quad T \cdot a(x) = T(a \cdot x),$$

where $a \in \mathcal{A}$, $x \in X$ and $T \in \mathfrak{B}(X,Y)$. This module is denoted by ${}_{\mathcal{A}}\mathfrak{B}(X,Y)$.

Let $\mathcal{A}$ be a Banach algebra and X be a Fréchet space then $\mathfrak{B}(\mathcal{A},X)$ is also Fréchet space with the family of semi-norms

$$Q_n(T) = \sup_{||x||\leq 1} |P_n(Tx)|,$$

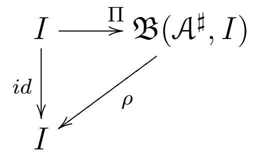
where $\{P_n\}_{n\in\mathbb{N}}$ is family of semi-norms of X. A Fréchet left $\mathcal{A}$-module X is called faithful if for $x \in X$ with $a \cdot x = 0$ for all $a \in \mathcal{A}$ we have $x = 0$. We denote unit linked of Banach algebra $\mathcal{A}$ with $\mathcal{A}^\sharp$.

For Fréchet spaces X and Y a morphism $\phi : X \to Y$ is said to be admissible if its kernel has, a topological direct complement in X and that the image is closed and also has a topological direct complement in Y.

Definition 2.1. *Let $\mathcal{A}$ be a Banach algebra, and let I be a Fréchet $\mathcal{A}$-module. Then I is called injective if, for every Fréchet $\mathcal{A}$-module X, Y and for each* *admissible monomorphism* $\rho \in_{\mathcal{A}} \mathfrak{B}(X, Y)$, *and for each* $\phi \in_{\mathcal{A}} \mathfrak{B}(X, I)$, *there exists* $\psi \in_{\mathcal{A}} \mathfrak{B}(Y, I)$ *such that* $\psi \circ \rho = \phi$.

Lemma 2.1. *Let* $\mathcal{A}$ *be a Banach algebra and* I *be a Fréchet* $\mathcal{A}$*-module. Then* I *is injective* $\mathcal{A}$*-module if and only if the left* $\mathcal{A}$*-module morphism* $\Pi \in_{\mathcal{A}} \mathfrak{B}(I, \mathfrak{B}(\mathcal{A}^{\sharp}, I))$ *has left inverse* $\rho \in_{\mathcal{A}} \mathfrak{B}(\mathfrak{B}(\mathcal{A}^{\sharp}, I), I)$, *where* Π *is the product map* $\Pi(x)(a) = a \cdot x$ *for all* $a \in \mathcal{A}^{\sharp}$ *and* $x \in I$.

Proof. Suppose that I is injective Fréchet $\mathcal{A}$-module. Then by the second part of the proof of [11, proposition 3.3] the product map $\Pi \in_{\mathcal{A}} \mathfrak{B}(I, \mathfrak{B}(\mathcal{A}^{\sharp}, I))$ is admissible. Hence by above definition there is a $\rho \in_{\mathcal{A}} \mathfrak{B}(\mathfrak{B}(\mathcal{A}^{\sharp}, I), I)$ such that the following diagram commute



On the other hand suppose there is a $\rho \in_{\mathcal{A}} \mathfrak{B}(\mathfrak{B}(\mathcal{A}^{\sharp}, I), I)$ such that $\rho \circ \Pi = id_I$ since by [11, proposition 3.3] $\mathfrak{B}(\mathcal{A}^{\sharp}, I)$ is injective so for Fréchet $\mathcal{A}$-modules E, F and admissible morphism$\gamma : E \to F$ and morphism $\phi : E \to I$ there is a morphism $\theta : E \to \mathfrak{B}(\mathcal{A}^{\sharp}, I)$ such that $\theta \circ \gamma = \Pi \circ \phi$. Now define $\psi = \rho \circ \theta$ then we get

$$\psi \circ \gamma = \rho \circ \phi \circ \gamma = \rho \circ \Pi \circ \phi = \phi$$

so I is injective. □

3. **Injectivity of $L^1(G, \omega)$ and $M(G, \omega)$**

Let G be a locally compact group. The map $\varphi_G : M(G, \omega) \to \mathbb{C}, \varphi_G(\mu) = \mu(G) = \langle \mu, 1 \rangle$ is called augmentation character. The augmentation character restricted to $L^1(G, \omega)$ has the form $\varphi_G(f) := \int_G f(x) dm(x), (f \in L^1(G, \omega))$. For a left Banach $L^1(G, \omega)$-module E an element $\lambda \in E'$ is called augmentation-invariant functional if $\langle f \cdot x, \lambda \rangle = \varphi_G(f) \langle x, \lambda \rangle, (f \in L^1(G, \omega), x \in E)$. E is called augmentation-invariant, if there exist a non-zero, augmentation invariant functional in E'.

Example 3.1. *For locally compact group* G *and weight function* ω, $M(G, \omega)$ *is augmentation-invariant with* $\lambda = \varphi_G$, *since* $\varphi_G(f \star \mu) = \varphi_G(f)\varphi_G(\mu)$ *where* $f \in L^1(G, \omega)$ *and* $\mu \in M(G, \omega)$. *Specially* $L^1(G, \omega)$ *so is.*
If λ *be the constant function* 1 *on* G, *regarded as an element of* $L^\infty(G, \frac{1}{\omega})$, *and hence as an element of* $E' = L^1(G, \omega)'''$, *then* λ *is an augmentation-invariant functional in* E', *since for* $\Lambda \in L^1(G, \omega)''$ *and* $f \in L^1(G, \omega)$ *we have*

$$\langle f \cdot \Lambda, 1 \rangle = \langle \Lambda, 1 \cdot f \rangle = \langle \Lambda, \varphi_G(f) 1 \rangle = \varphi_G(f) \langle \Lambda, 1 \rangle,$$

and so the module $L^1(G, \omega)''$ *is augmentation-invariant.*

For locally compact group G, an element Λ of $L^\infty(G)'$ is called mean, whenever $\langle 1, \Lambda\rangle = ||\Lambda|| = 1$. The group G is called amenable if there is a left-invariant mean on $L^\infty(G)$. We denote $P(G,\omega)$, as all $f \in L^1(G,\omega)$, such that $f \geq 0$ and $\langle 1, f\rangle = 1$. In special case when $\omega = 1$, we denote it by $P(G)$.

Proposition 3.1. *Let G be a locally compact group and ω be a weight function on G. Then the following statements are equivalent:*

(a) *There is a net $\{f_\alpha\} \subseteq P(G,\omega)$ such that $\lim_\alpha ||\delta_g \star f_\alpha - f_\alpha||_{1,\omega} = 0$ where δ_g is mass point at $g \in G$;*

(b) *G is amenable and ω is bounded.*

Proof. Let $\{f_\alpha\}$ be a net in $P(G,\omega)$ that satisfies (a). Since $P(G,\omega) \subset P(G)$, by [7, Proposition 0.8] there is an invariant mean on $L^\infty(G)$, and so G is amenable. Now for boundedness of ω, our proof is similar to that of [5]. In fact, let $E(\alpha) = \{s \in G : \Omega(s) < \alpha\}$ where $\Omega = \omega\tilde{\omega}$. Then for $h \in E(\beta)^c$ the complement of $E(\beta)$, let $x \in E(\alpha)h^{-1}$ then for some $y \in E(\alpha)h^{-1}$ we have $x = yh^{-1}$ so $\Omega(x) = \Omega(yh^{-1}) = \Omega(hy^{-1}) \geq \frac{\Omega(h)}{\Omega(y)} \geq \frac{\beta}{\alpha}$ and consequently

$$\chi_{E(\alpha)} \star \delta_h = \chi_{E(\alpha)h^{-1}} \leq \chi_{E(\frac{\beta}{\alpha})^c}, \tag{1}$$

where $h \in G$ and χ is characteristic function. In this case

$$\langle \chi_{E(\alpha)}, \Lambda\rangle = \langle \chi_{E(\alpha)} \star \delta_h, \Lambda\rangle \leq \langle \chi_{E(\frac{\beta}{\alpha})^c}, \Lambda\rangle \leq \frac{\alpha||\Lambda||}{\beta} \tag{2}$$

since

$$||\chi_{E(\frac{\beta}{\alpha})^c}||_{\infty,\omega} = \sup_{x\in G} \frac{\chi_{E(\frac{\beta}{\alpha})^c}(x)}{\omega(x)} = \sup_{x\in E(\frac{\beta}{\alpha})^c} \frac{1}{\omega(x)} \leq \frac{\alpha}{\beta}.$$

Suppose that ω is unbounded then Ω is unbounded and $E(\alpha)^c \neq \emptyset$ so for all $\alpha \in \mathbb{R}^+$ there is $h \in E(\alpha)^c$ such that the inequality (1) holds and so from (2) we have $\langle \chi_{E(\alpha)}, \Lambda\rangle = 0$ for all $\alpha \in \mathbb{R}^+$ thus,

$$\begin{aligned} 1 = \langle \chi_G, \Lambda\rangle &= \langle \chi_{E(\alpha)}, \Lambda\rangle + \langle \chi_{E(\alpha)^c}, \Lambda\rangle \\ &= \langle \chi_{E(\alpha)^c}, \Lambda\rangle. \end{aligned}$$

But by $\beta = 1$ and $\frac{1}{\alpha}$ instead of α in the relation (2) we have $\lim_\alpha \langle \chi_{E(\alpha)^c}, \Lambda\rangle = 0$ which is contradiction and hence Ω is bounded. Since $\omega \leq \omega\tilde{\omega} = \Omega$, so ω is bounded.

We note that if ω is bounded then $L^1(G) = L^1(G,\omega)$ so the converse is also true by [7, Proposition 0.8]. □

For a locally compact group G, set $A = L^1(G,\omega)$ and $\tilde{A} = L^1(G,\tilde{\omega})$. Then for $f \in A$ we define $f^\triangleleft \in \tilde{A}$ by the formula $f^\triangleleft(s) = f(s^{-1})\Delta_G(s^{-1})(s \in G)$, clearly the map $f \mapsto f^\triangleleft$ is a linear isometry from A into $\tilde{A}$, and we have $f^{\triangleleft\triangleleft} = f$ for $f \in A$ and $(f \star g)^\triangleleft = g^\triangleleft \star f^\triangleleft$ for $f, g \in A$. Furthermore

$\varphi_G(f^{\triangleleft}) = \varphi_G(f)$ for $f \in A$ and for all $T \in \mathfrak{B}(\tilde{A}, E)$, we define $T^{\triangleleft} \in \mathfrak{B}(A, E)$ by setting

$$T^{\triangleleft}(f) = T(f^{\triangleleft}), \quad f \in A.$$

Thus the map $T \mapsto T^{\triangleleft}$ is a linear isometry from $\mathfrak{B}(A, E)$ into $\mathfrak{B}(\tilde{A}, E)$ that is $||T|| = ||T^{\triangleleft}||$.

Theorem 3.1. *Let G be a locally compact group, $A = L^1(G, \omega)$ and E be the dual of the Banach right $\tilde{A}$-module F. Suppose that E is faithful and augmentation-invariant. Then E is injective if and only if G is amenable and ω is bounded.*

Proof. Let λ_0 be a non-zero augmentation-invariant functional on E. Then there is a $x_0 \in E$ with $\langle x_0, \lambda_0 \rangle = 1$, and set $T_0 = \Pi(x_0) \in \mathfrak{B}(\tilde{A}, E)$. From the lemma [4, proposition 1.7] there exists $\rho \in_{\tilde{A}} \mathfrak{B}(\mathfrak{B}(\tilde{A}, E), E)$ with $\rho \circ \Pi = I_E$ in particular $\rho(T_0) = x_0$ and hence also $T_0^{\triangleleft} \in \mathfrak{B}(A, E)$. After adjustment of λ_0 and x_0 by suitable non-zero constant we may suppose that $||T_0|| = ||T_0^{\triangleleft}|| = 1$ by considering the element $\lambda_0 \circ \rho \in \mathfrak{B}(\tilde{A}, E)'$ similarly to the proof of [4, lemma 4.3], we can find a left-invariant element $\Lambda_0 \in \mathfrak{B}(A, E)'$ such that $\langle T_0^{\triangleleft}, \Lambda_0 \rangle = 1$.

Now let $X = L^1(G, \omega)\hat{\otimes} F$, so from [4, lemma 4.4] there is a net $\{v_\alpha\}_{\alpha \in I}$ in X such that

$$\langle T_0^{\triangleleft}, v_\alpha \rangle = 1, \qquad \lim_\alpha ||L_s v_\alpha - v_\alpha||_\pi = 0, \tag{3}$$

where $(L_s v_\alpha)(t) = v_\alpha(s^{-1}t)$ and $\alpha \in I$, $s, t \in G$. By [10, Theorem 2.2] X is isomorphic with the space $L^1_\omega(G, F)$, that is, (the space of weighted F-valued integrable functions on G), so we can consider (v_α) as a net in $L^1_\omega(G, F)$ and hence $k_\alpha(t) = ||v_\alpha(t)||_F$, as a net in A. Since $L^1_\omega(G, F)$ and $L^1(G, F)$ are isometric (see [3, p. 66]) and $||T_0^{\triangleleft}|| = 1$ we have:

$$\begin{aligned}\langle k_\alpha, 1 \rangle &= \int_G k_\alpha(t) dm(t) \\ &= ||v_\alpha||_\pi = ||v_\alpha||_\pi^\omega \\ &\geq \langle T_0^{\triangleleft}, v_\alpha \rangle = 1\end{aligned}$$

where $||\cdot||_\pi$ and $||\cdot||_\pi^\omega$ are the norm of $L^1(G)\hat{\otimes} F$ and $L^1(G, \omega)\hat{\otimes} F$ respectively. Set $h_\alpha = k_\alpha / \langle k_\alpha, 1 \rangle$, so $\langle h_\alpha, 1 \rangle = 1$ and $h_\alpha \geq 0$, this shows that $h_\alpha \in P(G, \omega)$. Now take $s \in G$, we have $(L_s v_\alpha)(t) = v_\alpha(s^{-1}t)$ for $t \in G$ and so $(L_s h_\alpha)(t) = h_\alpha(s^{-1}t)$ for $t \in G$. It follows that

$$\begin{aligned}||L_s h_\alpha - h_\alpha||_{1,\omega} &\leq ||L_s k_\alpha - k_\alpha||_{1,\omega} \\ &\leq \int_G ||L_s v_\alpha - v_\alpha||_F \omega(t) dm(t) \\ &= ||L_s v_\alpha - v_\alpha||_\pi^\omega = ||L_s v_\alpha - v_\alpha||_\pi.\end{aligned}$$

Moreover from (3) we have $\lim_\alpha ||L_s h_\alpha - h_\alpha||_{1,\omega} = 0$, so the Proposition 3.1 implies that G is amenable and ω is bounded.

For the converse, suppose that G is amenable and ω is bounded then $L^1(G,\omega) = L^1(G,\tilde{\omega}) = L^1(G)$ so by [4, Theorem 4.6] E is injective, which completes the proof. □

Corollary 3.1. *Let G be a locally compact group and ω be a weight function on G. Then the following statements are equivalent:*

(a) *The group G is amenable and ω is bounded;*
(b) *$M(G,\omega)$ is injective as a Banach $L^1(G,\omega)$-module;*
(c) *$L^1(G,\omega)''$ is injective as a Banach $L^1(G,\omega)$-module.*

Proof. Apply the previous theorem and example 3.1 to the dual Banach right $L^1(G,\tilde{\omega})$-modules $M(G,\omega)$ and $L^1(G,\omega)''$. □

As an another application of the last theorem, we characterise injectivity of the class of Fréchet algebras. Suppose that $\{\omega_n\}$ is an increasing sequence of weight functions with condition $\omega_n \geq 1$ for all $n \in \mathbb{N}$. Then the families $\{L^1(G,\omega_n)\}$ and $\{M(G,\omega_n)\}$ are decreasing sequences of Banach algebrs. Now we define $A(\omega) = \cap_1^\infty L^1(G,\omega_n), B(\omega) = \cap_1^\infty M(G,\omega_n)$. These algebras are projective limit of Banach algebras $L^1(G,\omega_n)$ and $M(G,\omega_n)$, respectively, so these are Fréchet algebras, for more details see [9].

Theorem 3.2. *Let G be a locally compact group and $\{\omega_n\}_{n=0}^\infty$ be a sequence of increasing weight functions on G with condition $1 \leq \omega_n \leq \omega_0$ for all $n \in \mathbb{N}$. Then the following statements hold:*

(i) *$A(\omega)$ is injective as $L^1(G,\omega_0)$-module if and only if G is amenable, discrete and ω_0 is bounded.*
(ii) *$B(\omega)$ is injective as $L^1(G,\omega_0)$-module if and only if G is amenable and ω_0 is bounded.*

Proof. (i) Let $\mathcal{A} = L^1(G,\omega_0)$ and let $A(\omega)$ be injective $\mathcal{A}$-module. By Lemma 2.1 the map $\Pi \in_{\mathcal{A}} \mathfrak{B}(A(\omega), \mathfrak{B}(\mathcal{A}^\sharp, A(\omega)))$ has a left inverse $\rho \in \mathfrak{B}(\mathfrak{B}(\mathcal{A}^\sharp, A(\omega)), A(\omega))$. If we denote $\overline{A(\omega)} = \overline{A(\omega)}^{||\cdot||_{1,\omega_n}}$ then $\rho \in \mathfrak{B}(\mathfrak{B}(\mathcal{A}^\sharp, A(\omega)), \overline{A(\omega)})$, because $\overline{A(\omega)} = L^1(G,\omega_n)$ and the topology of $L^1(G,\omega_n)$ is coarsest than the topology of $A(\omega)$. Thus consider $\overline{\Pi} \in_{\mathcal{A}} \mathfrak{B}(\overline{A(\omega)}, \mathfrak{B}(\mathcal{A}, \overline{A(\omega)}))$ as the product map and $\bar{\rho} \in \mathfrak{B}(\mathfrak{B}(\mathcal{A}, \overline{A(\omega)}), \overline{A(\omega)})$ as extension of ρ and let $x \in L^1(G,\omega_n)$ and net $\{x_\alpha\} \subset A(\omega)$ converges to x. Then we have

$$\begin{aligned} ||\bar{\rho} \circ \overline{\Pi}(x) - x_\alpha||_{1,\omega_n} &= ||\bar{\rho} \circ \overline{\Pi}(x) - \bar{\rho} \circ \overline{\Pi}(x_\alpha)||_{1,\omega_n} \\ &\leq ||\bar{\rho}||||x - x_\alpha||_{1,\omega_n}, \end{aligned}$$

this means that $\bar{\rho} \circ \overline{\Pi}(x) = x$ for all $x \in L^1(G,\omega_n)$. Thus Lemma 2.1 shows that $L^1(G,\omega_n)$ is an injective $\mathcal{A}$-module and also from Theorem 3.1 ω_n is bounded and hence $L^1(G,\omega_n) = L^1(G)$, so [4, Theorem 4.9] implies that G is amenable and discrete.

(ii) Since $B(\omega)$ is *so*-dense in $M(G,\omega_n)$, so for all $\mu \in M(G,\omega_n)$ there is a net $\{\mu_\alpha\} \subset B(\omega)$ such that $||a\cdot\mu - a\cdot\mu_\alpha||_{\omega_n} \to 0$ for all $a \in L^1(G,\omega_0)$. Similarly to (i), if $B(\omega)$ replaced by $A(\omega)$ we have

$$\begin{aligned} ||\bar{\rho}\circ\overline{\Pi}(a\cdot\mu) - a\cdot\mu_\alpha||_{1,\omega_n} &= ||\bar{\rho}\circ\overline{\Pi}(a\cdot\mu) - \bar{\rho}\circ\overline{\Pi}(a\cdot\mu_\alpha)||_{1,\omega_n} \\ &\leq ||\bar{\rho}||||a\cdot\mu - a\cdot\mu_\alpha||_{1,\omega_n}, \end{aligned}$$

that is $\bar{\rho}\circ\overline{\Pi}(a\cdot\mu) = a\cdot\mu$. Now from Cohen's factorization theorem [1, Corollary 2.9.26] for all $\mu \in M(G,\omega_n)$ there is an $a \in L^1(G,\omega_0)$ and $\nu \in M(G,\omega_n)$ such that $\mu = a\cdot\nu$. Thus

$$\bar{\rho}\circ\overline{\Pi}(\mu) = \bar{\rho}\circ\overline{\Pi}(a\cdot\nu) = a\cdot\nu = \mu,$$

which implies that $M(G,\omega_n)$ is injective $L^1(G,\omega_0)$-module. Then, by corollary 3.1, G is amenable and ω is bounded. This completes the proof of the theorem. □

Corollary 3.2. *Let G be a locally compact group and ω is weight function. Then $L^1(G,\omega)$ is injective Banach $L^1(G,\omega)$-module if and only if G is discrete, amenable and ω is bounded.*

By the same assumption of the theorem 3.2, the Fréchet $L^1(G,\omega_0)$-module $A(\omega)$ is injective Fréchet $L^1(G,\omega_0)$-module if and only if $L^1(G,\omega_n)$ is injective Banach $L^1(G,\omega_0)$-module for all $n \in \mathbb{N}$. And similarly $B(\omega)$ as a Fréchet $L^1(G,\omega_0)$-module is injective if and only if $M(G,\omega_n)$ is Banach $L^1(G,\omega_0)$-module for all $n \in \mathbb{N}$.

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