THE NORMS AND THE LOWER BOUNDS FOR MATRIX OPERATORS ON WEIGHTED DIFFERENCE SEQUENCE SPACES

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This paper is concerned with the problem of finding upper bounds and lower bounds of matrix operators from \( l_p(v) \) into \( l_p(w, \Delta) \), where \((v_n)\) and \((w_n)\) are two non-negative sequences. Moreover, the norms and lower bounds of matrix operators such as quasi-summability matrices and Hilbert operator are computed.

Keywords: Matrix operator, Norm, Lower bound, Quasi-summability matrix, Hilbert matrix, Weighted sequence space.

MSC2010: 26D15, 40C05, 40G05, 47B37.

1. Introduction

Let \( p \geq 1 \) and \( \omega \) denote the set of all real-valued sequences. The space \( l_p \) is the set of all real sequences \( x = (x_n) \in \omega \) such that
\[
\|x\|_p = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < \infty.
\]
If \( w = (w_n) \in \omega \) is a non-negative sequence, we define the weighted sequence space \( l_p(w) \) as follows:
\[
l_p(w) := \left\{ x = (x_n) \in \omega : \sum_{n=1}^{\infty} w_n |x_n|^p < \infty \right\},
\]
with norm, \( \|\cdot\|_{p,w} \), which is defined in the following way:
\[
\|x\|_{p,w} = \left( \sum_{n=1}^{\infty} w_n |x_n|^p \right)^{1/p}.
\]
The idea of difference sequence spaces was introduced by Kizmaz [9]. Similarly, we define the sequence space \( l_p(w, \Delta) \) as below:
\[
l_p(w, \Delta) = \left\{ x = (x_n) \in \omega : \sum_{n=1}^{\infty} w_n |x_n - x_{n+1}|^p < \infty \right\},
\]
with semi-norm, \( \|\cdot\|_{p,w,\Delta} \), which is defined by
\[
\|x\|_{p,w,\Delta} = \left( \sum_{n=1}^{\infty} w_n |x_n - x_{n+1}|^p \right)^{\frac{1}{p}}.
\]

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Let bounded operator from $l^p(w, \Delta)$ possible value of $l^p$. In particular if Lyons [11], and has been intensively studied on for all non-negative decreasing sequence $x$. If $\sup x \in l^p(v)$, then $A$ is an operator from $l^p(v)$ into $l^p(w, \Delta)$, and $\|A\|_{p,w,\Delta}$ for the norm of $A$ as an operator from $l^p(w, \Delta)$ into $l^p(w, \Delta)$, and $\|A\|_{p,\Delta}$ for the norm of $A$ as an operator from $l^p$ into $l^p(\Delta)$, and $\|A\|_p$ for the norm of $A$ as an operator from $l^p$ into itself.

The problem of finding the upper bounds of certain matrix operators on the sequence spaces $l^p(w, \Delta)$, $d(w, p)$ and $b_{v}p$ are studied before in [5], [6], [8] and [10]. In the study, we examine this problem for matrix operators from $l^p(v)$ into $l^p(w, \Delta)$ and we consider certain matrix operators such as quasi-summability matrices and Hilbert operator.

Let $A$ be a matrix operator with non-negative entries from $l^p(v)$ into $l^p(w, \Delta)$. The other purpose of this study is to consider the inequality of the form

$$\|Ax\|_{p,w,\Delta} \geq L\|x\|_{p,v},$$

for all non-negative decreasing sequence $x \in l^p(v)$, where $L$ is a constant not depending on $x$. Also we seek the largest possible value of $L$.

The problem of finding the lower bounds of matrix operators was introduced by Lyons [11], and has been intensively studied on $l^p$ by Bennett [1,2,3]. Jameson [6] was computed the lower bounds of operators on Lorentz sequence space $d(w, 1)$. Then Jameson and Lashkaripour [7] were examined lower bounds of certain matrix operators on $l^p(w)$ and $d(w, p)$. More recently, this problem has been developed in the block sequence space [5]. In this paper, we study the problem of finding the lower bound for matrix operators from $l^p(v)$ into $l^p(w, \Delta)$ and investigate certain matrix operators such as quasi-summability matrices and Hilbert operator.

2. The norm of matrix operators from $l^1(v)$ into $l_1(w, \Delta)$

In this section, we tend to compute the norm of operators from $l_1(v)$ into $l_1(w, \Delta)$. We may begin with the following theorem which is essential in the study.

**Theorem 2.1.** Let $A = (a_{n,k})$ be a matrix operator and $(v_n)$, $(w_n)$ be two non-negative sequences. If $\sup k u_k < \infty$ where $u_k = \sum_{n=1}^{\infty} w_n |a_{n,k} - a_{n+1,k}|$ for all $k$, then $A$ is a bounded operator from $l_1(v)$ into $l_1(w, \Delta)$ and $\|A\|_{1,v,w,\Delta} = \sup_n \frac{u_n}{v_n}$.

In particular if $v_n = w_n = 1$ for all $n$, then $A$ is a bounded operator from $l_1$ into $l_1(\Delta)$ and $\|A\|_{1,1,\Delta} = \sup_n u_n$. 
Proof. Let \( M = \sup_n \frac{u_n}{v_n} \) and \((x_n)\) be in \( l_1(v)\). We have
\[
\|Ax\|_{1,w,\Delta} = \sum_{n=1}^{\infty} w_n \left( \sum_{k=1}^{\infty} (a_{n,k} - a_{n+1,k})x_k \right) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} w_n |a_{n,k} - a_{n+1,k}| |x_k|
\]
\[
= \sum_{k=1}^{\infty} u_k |x_k| \leq M \sum_{k=1}^{\infty} v_k |x_k| = M \|x\|_{1,v},
\]
which says that \( \|A\|_{1,v,w,\Delta} \leq M \). Conversely, we take \( x = e_n \) which \( e_n \) denotes the sequence having 1 in place \( n \) and 0 elsewhere, then \( \|x\|_{1,v} = v_n \) and \( \|Ax\|_{1,w,\Delta} = u_n \) which proves that \( \|A\|_{1,v,w,\Delta} = M \).

We say that \( A = (a_{n,k}) \) is a quasi-summability matrix if it is an upper-triangular matrix, i.e. \( a_{n,k} = 0 \) for \( n > k \), and \( \sum_{n=1}^{k} a_{n,k} = 1 \) for all \( k \). In the following, we consider the norm problem for quasi-summability matrix operators.

**Theorem 2.2.** Let \( A = (a_{n,k}) \) be an upper-triangular matrix with non-negative entries and \((w_n)\) be an increasing sequence. If the columns of \( A \) are decreasing, i.e.
\[
a_{n,k} \geq a_{n+1,k}, \quad (n,k = 1, 2, \ldots)
\]
and \( M = \sup_k a_{1,k} < \infty \), then \( A \) is a bounded operator from \( l_1(w) \) into \( l_1(w;\Delta) \) and
\[
\|A\|_{1,w,\Delta} \leq M.
\]
In particular if \( A \) is a quasi-summability matrix, then \( \|A\|_{1,w,\Delta} = 1 \).

**Proof.** According to above notation
\[
u_k = \sum_{n=1}^{k-1} w_n (a_{n,k} - a_{n+1,k}) + w_k a_{k,k},
\]
Since the sequences \((w_n)\) is increasing
\[
u_k \leq \sum_{n=1}^{k-1} (a_{n,k} - a_{n+1,k}) + a_{k,k} = a_{1,k},
\]
so by applying Theorem 2.1, we have \( \|A\|_{1,w,\Delta} \leq M \). In particular if \( A \) is a quasi-summability matrix, we deduce that \( \|A\|_{1,w,\Delta} \leq 1 \). Using the fact that \( \|Ae_1\|_{1,w,\Delta} = \|e_1\|_{1,w} = w_1 \) finishes the proof.

Next, we identify a class of quasi-summability matrices for which the norm problem is very easy. If \( (a_n) \) is a non-negative sequence with \( a_1 > 0 \) and \( A_n = a_1 + \cdots + a_n \), the matrix \( M_a = (a_{n,k}) \) is defined as follows:
\[
a_{n,k} = \begin{cases} 
\frac{a_n}{M_a} & n \leq k \\
0 & n > k.
\end{cases}
\]
(1)

\( M_a \) is the transpose of the weighted mean matrix.

**Corollary 2.1.** If \( (a_n) \) is decreasing and \((w_n)\) is increasing, then \( M_a \) is a bounded operator from \( l_1(w) \) into \( l_1(w;\Delta) \) and
\[
\|M_a\|_{1,w,\Delta} = 1.
\]
Note that $M_\alpha$ is called the Copson matrix when $a_n = 1$ for all $n$, hence the Copson Matrix $C = (c_{n,k})$ defined by
\[
c_{n,k} = \begin{cases} 
  \frac{1}{k} & \text{for } n \leq k \\
  0 & \text{for } n > k.
\end{cases}
\]

**Corollary 2.2.** Let $C$ be the Copson operator and $(v_n)$ and $(w_n)$ be two non-negative sequences. If $\sup_k \frac{w_k}{k^{\alpha}} < \infty$, then $C$ is a bounded operator from $l_1(v)$ into $l_1(w, \Delta)$ and
\[
\left\|C\right\|_{1,v,w,\Delta} = \sup_k \frac{w_k}{k^{\alpha}}.
\]

**Proof.** Since
\[
 u_k = \sum_{n=1}^{\infty} w_n (c_{n,k} - c_{n+1,k}) = \frac{w_k}{k},
\]
by applying Theorem 2.1, we obtain the desired result. 

In the next statement, we try to compute the norm of the certain matrix operators from $l_1$ into $l_1(\Delta)$.

**Theorem 2.3.** Suppose that $A = (a_{n,k})$ is a matrix with non-negative entries and $M = \sup_a a_{1,k} < \infty$. If the columns of $A$ are decreasing i.e.
\[
a_{n,k} \geq a_{n+1,k}, \quad (n, k = 1, 2, \cdots)
\]
and $\lim_{n \to \infty} a_{n,k} = 0$, for all $k$. Then $A$ is a bounded operator from $l_1$ into $l_1(\Delta)$ and
\[
\|A\|_{1,\Delta} = M.
\]
In particular if $A$ is a quasi-summability matrix, then $\|A\|_{1,\Delta} = 1$.

**Proof.** Since
\[
 u_k = \sum_{n=1}^{\infty} (a_{n,k} - a_{n+1,k}) = a_{1,k},
\]
by using Theorem 2.1, we obtain the desired result.

We recall the Hilbert operator $H$ which is defined by the matrix:
\[
h_{n,k} = \frac{1}{n+k}, \quad (n, k = 1, 2, \cdots).
\]

**Corollary 2.3.** If $H$ is the Hilbert operator, then $H$ is a bounded operator from $l_1$ into $l_1(\Delta)$ and $\|H\|_{1,\Delta} = \frac{\pi}{2}$.

In the following, we try to solve the problem of finding the norm of the Hilbert matrix operator from $l_1(v)$ into $l_1(w, \Delta)$. For this purpose, the same as the most studies of the Hilbert operator, it uses the well-known integral
\[
\int_{0}^{\infty} \frac{1}{t^\alpha (t+e)} dt = \frac{\pi}{e^\alpha \sin \alpha \pi},
\]
where $0 < \alpha < 1$, (see [4], page 285).

**Theorem 2.4.** Let $H$ be the Hilbert operator. If $w_n = \frac{1}{n^\alpha}$ for all $n$, where $0 < \alpha < 1$, then $H$ is a bounded operator from $l_1(w)$ into $l_1(w, \Delta)$ and
\[
\|H\|_{1,w,\Delta} \leq \frac{\pi}{\sin \alpha \pi} \left(1 - \frac{1}{2\alpha}\right).
\]
According to above notation

\[
\begin{align*}
  u_n &= \sum_{i=1}^{\infty} \frac{1}{i^{\alpha}} \left( \frac{1}{i + n} - \frac{1}{i + n + 1} \right) \\
  &= \frac{\pi}{\sin \alpha \pi} \left( \frac{1}{n^{\alpha}} - \frac{1}{(n + 1)^{\alpha}} \right),
\end{align*}
\]

so

\[
\begin{align*}
  n^{\alpha} u_n &\leq \frac{\pi}{\sin \alpha \pi} \left( 1 - \left( \frac{n}{n + 1} \right)^{\alpha} \right) \\
  &\leq \frac{\pi}{\sin \alpha \pi} \left( 1 - \frac{1}{2^{\alpha}} \right),
\end{align*}
\]

hence \( \|H\|_{1,w,\Delta} \leq \frac{\pi}{\sin \alpha \pi} \left( 1 - \frac{1}{2^{\alpha}} \right). \)

\[ \square \]

3. Upper bounds of matrix operators from \( l_p(v) \) into \( l_p(w, \Delta) \)

In this section the problem of finding the norm of certain matrix operators such as the transpose of the weighted mean, Copson and Hilbert from \( l_p(v) \) into \( l_p(w, \Delta) \) are considered. We first give the Schur’s Theorem and a lemma which are essential in the study.

**Lemma 3.1** ([8], Lemma 2.2). Let \( p > 1 \) and \( B = (b_{n,k}) \) be a matrix operator with \( b_{n,k} \geq 0 \) for \( n, k = 1, 2, \ldots \). Suppose that \((s_n)\) and \((t_k)\) are two sequences of strictly positive numbers such that for some \( c, R \)

\[
s_n^{1/p} \sum_{k=1}^{\infty} b_{n,k} t_k^{1/p} \leq R \quad \text{(for } n \geq 1), \quad t_k^{(p-1)/p} \sum_{n=1}^{\infty} b_{n,k} s_n^{1-p/p} \leq C \quad \text{(for } k \geq 1). 
\]

Then \( \|B\|_p \leq R^{(p-1)/p} C^{1/p} \).

**Lemma 3.2.** Let \( p \geq 1 \) and \((v_n), (w_n)\) be two non-negative sequences. If \( A = (a_{n,k}) \) and \( B = (b_{n,k}) \) are two matrix operators such that \( b_{n,k} = \left( \frac{w_n}{v_k} \right)^{1/p} (a_{n,k} - a_{n+1,k}) \), then

\[
\|A\|_{p,v,w,\Delta} = \|B\|_p.
\]

Hence, if \( B \) is a bounded operator on \( l_p \), then \( A \) will be a bounded operator from \( l_p(v) \) into \( l_p(w, \Delta) \).

**Proof.** For every \( x \in l_p(v) \), we define \( y = (y_k) \) by \( y_k = v_k^{1/p} x_k \). It is obvious that \( \|x\|_{p,v} = \|y\|_p \), and

\[
\begin{align*}
  \|A\|_{p,v,w,\Delta} &= \sup_{x \in l_p(v)} \frac{\|Ax\|_{p,w,\Delta}}{\|x\|_{p,v}} = \sup_{x \in l_p(v)} \sum_{n=1}^{\infty} w_n \left| \sum_{k=1}^{\infty} (a_{n,k} - a_{n+1,k}) x_k \right|^p \\
  &= \sup_{y \in l_p} \left( \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \frac{w_n^{1/p} (a_{n,k} - a_{n+1,k}) y_k}{v_k^{1/p}} \right)^p \right)^{1/p} \leq \sup_{y \in l_p} \frac{\|By\|_p}{\|y\|_p} = \|B\|_p.
\end{align*}
\]

\[ \square \]

In the following, we investigate the norm of the transpose of the weighted mean matrix.
Let $\|M_n\|_{p,v,1,\Delta} = 1$. Then Theorem 3.1.

Proof. By applying Lemma 3.2 we have $\|M_n\|_{p,v,1,\Delta} = \|M\|_p$, where

$\begin{align*}
b_{n,k} &= \begin{cases} 
\frac{a_k(a_n-a_{n+1})}{A_k} & \text{for } n < k \\
\frac{a_k}{A_k} & \text{for } n = k \\
0 & \text{for } n > k.
\end{cases}
\end{align*}$

In Lemma 3.1, take $s_n = t_n = 1$ and let $C, R$ be defined as before. Since

$\begin{align*}
\sum_{k=1}^{\infty} b_{n,k} &= \frac{a_n^2}{A_{n-1}A_n} + \sum_{k=n+1}^{\infty} \frac{a_k(a_n-a_{n+1})}{A_k} \\
&= \frac{a_n^2}{A_{n-1}A_n} + \frac{(a_n-a_{n+1})}{A_{n-1}A_n} \sum_{k=n+1}^{\infty} \left( \frac{1}{A_{n-1}} - \frac{1}{A_k} \right) \\
&= \frac{a_n^2}{A_{n-1}A_n} + \frac{(a_n-a_{n+1})}{A_{n-1}A_n},
\end{align*}$

we have $\sum_{k=1}^{\infty} b_{1,k} = 1$ and

$\sum_{k=1}^{\infty} b_{n,k} = \frac{a_n}{A_{n-1}} - \frac{a_{n+1}}{A_{n}}$

for $n > 1$, so $R \leq 1$. Also, since

$\begin{align*}
\sum_{n=1}^{\infty} b_{n,k} &= \sum_{n=1}^{k-1} \frac{a_k(a_n-a_{n+1})}{A_{k-1}A_k} + \frac{a_k^2}{A_{k-1}A_k} \\
&= \frac{a_k}{A_{k-1}A_k} \leq 1,
\end{align*}$

we deduce that $C \leq 1$, so $\|M_n\|_{p,v,1,\Delta} \leq 1$. Now let $x = (1, 0, 0, \cdots)$, we have $\|x\|_{p,v} = 1$ and $\|M_nx\|_{p,\Delta} = 1$. So $\|M_n\|_{p,v,1,\Delta} \geq 1$, this completes the proof of the theorem. $\square$

Now we are ready to compute the norm of the Copson matrix operator.

Theorem 3.2. Suppose that $p > 1$ and $(v_n)$, $(w_n)$ are two non-negative sequences. If $C$ is the Copson matrix operator and

$\begin{align*}
M &= \sup_n \frac{1}{n} \left( \frac{w_n}{v_n} \right)^{1/p} < \infty,
\end{align*}$

then $\|C\|_{p,v,w,\Delta} = M$. In particular if $v_n = w_n$ for all $n$, we have $\|C\|_{p,v,\Delta} = 1$.

Proof. By applying Lemma 3.2 we have $\|C\|_{p,v,w,\Delta} = \|B\|_p$, where

$\begin{align*}
b_{n,k} &= \begin{cases} 
\frac{1}{n} \left( \frac{w_n}{v_n} \right)^{1/p} & \text{for } n = k \\
0 & \text{otherwise}.
\end{cases}
\end{align*}$

In Lemma 3.1, take $s_n = t_n = 1$ and let $C, R$ be defined as before. Since the matrix $B$ is diagonal, we deduce that $R \leq M$ and $C \leq M$, so $\|C\|_{p,v,w,\Delta} \leq M$. Now let $x = e_n$, we have $\|x\|_{p,v} = v_n^{1/p}$ and $\|Cx\|_{p,w,\Delta} = \frac{w_n^{1/p}}{n}$. So $\|C\|_{p,v,w,\Delta} \geq M$, which concludes the proof. $\square$

Finally, we consider the norm of the Hilbert matrix operator.
Let Suppose By applying Lemma 3.2 we have The proof is essentially same as that of Theorem 3.3 and so we omit the details.

Also let Then that Theorem 4.1.

Lemma 4.1 for all non-negative decreasing sequence \( x \) where \( L \) \( A \) \( M \) \( H \). If \( A_n = \sum_{i=1}^{\infty} a_i x_i \) is convergent, then

\[
\left( \sum_{i=1}^{\infty} a_i x_i \right)^{\frac{p}{q}} \geq \sum_{n=1}^{\infty} A_n^{\frac{p}{q}} (x_n^p - x_{n+1}^p).
\]

4. Lower bounds of matrix operators from \( l_p(v) \) into \( l_p(w, \Delta) \)

In this part of the study, we consider the lower bound, \( L \), of the form

\[
\|Ax\|_{p, w, \Delta} \geq L \|x\|_{p, v},
\]

for all non-negative decreasing sequence \( x \). The constant \( L \) is not depending on \( x \) and we seek the largest possible value of \( L \). We are looking for the problem of finding the lower bound of certain matrix operators from \( l_p(v) \) into \( l_p(w, \Delta) \).

We begin with a lemma, which is the key to prove the main theorem of this section.

Lemma 4.1 ([7], Lemma 2). Suppose \( p \geq 1 \) and sequences \( (a_i) \) and \( (x_i) \) are nonnegative, and that \( (x_i) \) is decreasing and tends to zero. If \( A_n = \sum_{i=1}^{\infty} a_i \), \( A_0 = 0 \) and \( B_n = \sum_{i=1}^{n} a_i x_i \), then

\[
(i) \quad B_n^p - B_{n-1}^p \geq (A_n^p - A_{n-1}^p)x_n^p \text{ for all } n;
(ii) \quad \text{if } \sum_{i=1}^{\infty} a_i x_i \text{ is convergent, then}
\]

\[
\left( \sum_{i=1}^{\infty} a_i x_i \right)^{\frac{p}{q}} \geq \sum_{n=1}^{\infty} A_n^{\frac{p}{q}} (x_n^p - x_{n+1}^p).
\]

Theorem 4.1. Let \( p \geq 1 \) and \( (v_n), (w_n) \) be non-negative sequences, and that \( \sum_{n=1}^{\infty} v_n = \infty \). Also let \( A = (a_{n,k}) \) be a matrix operator from \( l_p(v) \) into \( l_p(w, \Delta) \) such that \( a_{n,k} \geq a_{n+1,k} \) for all \( n, k \). If \( V_n = \sum_{k=1}^{n} v_k \) and \( S_n = \sum_{i=1}^{n} \left( \sum_{k=1}^{n} (a_{i,k} - a_{i+1,k}) \right)^p \) then

\[
\|Ax\|_{p, w, \Delta} \geq L \|x\|_{p, v},
\]
for all non-negative decreasing sequence \( x \in l_p(v) \), where

\[
L^p = \inf_n \frac{S_n}{V_n}.
\]

This constant is the best possible.

Proof. Let \( x \) be in \( l_p(v) \) such that \( x_1 \geq x_2 \geq \cdots \geq 0 \) and \( \|x\|_{p,v} = 1 \). The condition \( \sum_{n=1}^{\infty} v_n = \infty \) implies that \( \lim_{n \to \infty} x_n = 0 \). Applying Lemma 4.1 and Able’s identity, we have

\[
\|Ax\|_{p,w,\Delta}^p = \sum_{n=1}^{\infty} w_n \left( \sum_{k=1}^{\infty} (a_{n,k} - a_{n+1,k}) x_k \right)^p \\
\geq \sum_{n=1}^{\infty} w_n \sum_{i=1}^{\infty} \left( \sum_{k=1}^{i} (a_{n,k} - a_{n+1,k}) \right) (x_i^p - x_{i+1}^p) \\
= \sum_{n=1}^{\infty} S_i (x_i^p - x_{i+1}^p) \geq L^p \sum_{i=1}^{\infty} V_i (x_i^p - x_{i+1}^p) = L^p \|x\|_{p,v}.
\]

So \( \|Ax\|_{p,w,\Delta} \geq L^p \|x\|_{p,v} \). To show that the above constant is the best possible, we take \( x_1 = x_2 = \cdots = x_n = 1 \), and \( x_k = 0 \) for all \( k \geq n+1 \), then \( \|x\|_{p,v} = V_n \) and \( \|Ax\|_{p,w,\Delta} = S_n \), which finishes the proof of the theorem. \( \square \)

The problem of finding lower bound for the Copson matrix operator for certain weights is solved in the following.

**Theorem 4.2.** Let \( p \geq 1 \) and \( (v_n) \), \( (w_n) \) be non-negative sequences, and that \( \sum_{n=1}^{\infty} v_n = \infty \). If \( C \) is the Copson operator, then

\[
\|Cx\|_{p,w,\Delta} \geq L \|x\|_{p,v},
\]

for all non-negative decreasing sequence \( x \), where

\[
L^p = \inf_n \frac{w_1 + \frac{w_2}{p} + \cdots + \frac{w_p}{p^p}}{V_n}.
\]

In particular

1. if \( V_n = n^{p+1} \) and \( w_n = n^{2p} \) for all \( n \), then \( L^p = \frac{1}{p+1} \);
2. if \( v_n = 1 \) and \( w_n = n^p \) for all \( n \), then \( L = 1 \);
3. if \( v_n = w_n \) for all \( n \), then \( L = 0 \).

Proof. With the notation of Theorem 4.1, \( S_n = \sum_{k=1}^{n} \frac{w_k}{k^p} \) which completes the proof of the first part. If \( w_n = n^{2p} \) and \( V_n = n^{p+1} \) then

\[
L^p = \lim_{n \to \infty} \left( \frac{1}{n} \right)^p + \left( \frac{2}{n} \right)^p + \cdots + \left( \frac{n}{n} \right)^p \frac{n}{n} = \int_{0}^{1} x^p dx = \frac{1}{p+1}.
\]

The remaining of the proof is obvious. \( \square \)

Write \( t_n = \sum_{i=1}^{\infty} w_i (a_{i,n} - a_{i+1,n})^p \), since \( s_n = S_n - S_{n-1} \) we have the following statement.

**Proposition 4.1.** If \( v \), \( w \) and \( A \) satisfy the conditions of Theorem 4.1, and

\[
a_{n,k} - a_{n+1,k} \geq a_{n,k+1} - a_{n+1,k+1},
\]

for \( n, k = 1, 2, \cdots \), then

\[
L^p \geq \inf_n [n^p - (n-1)^p] \frac{t_n}{w_n}.
\]
Proof. See Proposition 1 of [7].

Finally, we compute the lower bound of Hilbert operator from $l_p(w)$ into $l_p(\Delta, w)$.

**Theorem 4.3.** Suppose that $w_n = \frac{1}{n}$ and $v_n = \frac{1}{n^{p+\alpha}}$ for all $n$, where $0 \leq p + \alpha \leq 1$ and $p \geq 1$. If $H$ is the Hilbert operator, then

$$\|Hx\|_{p,w,\Delta} \geq L\|x\|_{p,v},$$

for all non-negative decreasing sequence $x$, where

$$L^p = \sum_{i=1}^{\infty} \frac{1}{i^n(i+1)^p(i+2)^p}.$$

**Proof.** We have

$$L^p \geq \inf_n \frac{n^p - (n-1)^p}{v_n} \geq \inf_n \frac{n^{p-1}}{v_n} \geq \inf_n \frac{n^{2p+\alpha-1}}{v_n} \sum_{i=1}^{\infty} w_i(h_{i,n} - h_{i+1,n})^p,$$

Now let $E_k = \{i \in \mathbb{N} : (k-1)n \leq i \leq kn\}$ for $k = 1, 2, \ldots$. For $i \in E_k$, we have

$$\left(\frac{i}{n}\right)^\alpha (i+n)^p(i+n+1)^p \leq k^n n^{2p}(k+1)^p(k+2)^p,$$

so

$$\sum_{i \in E_k} \left(\frac{i}{n}\right)^\alpha (i+n)^p(i+n+1)^p \geq \frac{n}{k^n n^{2p}(k+1)^p(k+2)^p}.$$

Hence

$$L^p \geq \sum_{k=1}^{\infty} \frac{1}{k^n(k+1)^p(k+2)^p}.$$

Since $\|e_1\|_{p,v} = 1$ and

$$\|He_1\|_{p,w,\Delta} = \sum_{n=1}^{\infty} \frac{1}{n^n(n+1)^p(n+2)^p},$$

and also $L = \inf_{x \in l_p(v)} \frac{\|Hx\|_{p,w,\Delta}}{\|x\|_{p,v}} \leq \frac{\|He_1\|_{p,w,\Delta}}{\|e_1\|_{p,v}}$, which concludes the proof.

**REFERENCES**


