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FIXED POINTS FOR A CLASS OF EXTENDED INTERPOLATIVE ψF-CONTRACTION MAPS OVER A b-METRIC SPACE AND ITS APPLICATION TO DYNAMICAL PROGRAMMING

Sayantan PANJA[1], Kushal ROY[2], Mantu SAHA[3]

This article deals with a class of extended interpolative Reich-Rus type ψF-contraction mapping and the existence of their fixed points over b-metric spaces. Several examples have been given in support of our established theorem. Also an example has been presented here to show that our proposed contractive mapping is a real generalization of ψF-contraction mappings. Moreover, an application is given to ensure the existence of solutions for some functional equations relating to the problem of dynamical programming.

Keywords: b-metric space, Fixed Point, Extended Interpolative Reich-Rus type ψF-contraction, Dynamical Programming.

1. Introduction

The study of fixed point theory is one of the important area of research in Mathematical Sciences. The study of mappings together with their characterizations for searching of their fixed points is a central topic in fixed point theory. The most celebrated work in this area goes back to 1922 due to Polish Mathematician S. Banach [4] and the result is known as *Banach Contraction Principle*. The fixed point theory is applicable for solving differential equations, integral equations, system of linear algebraic equations and also in many diversified areas namely optimal control theory, probabilistic theory, homotopy theory, approximation theory, optimization theory, mathematical economics and the like. Following the discovery of Banach Contraction Principle, the area of fixed point theory, particularly in metric fixed point theory had been flooded by the works of many researchers who had been able to generalize Banach contraction principle in different directions. On examining the utilities of fixed point theorems along with their extended versions, one can see that extensions could be done on the choice of mappings and the underlying spaces in which such class of mappings had been used (we may refer to [9], [11], [15], [23], [25], [26]).

In 2018, E. Karapinar[16] introduced a new type of contractive mapping via interpolation, called Interpolative Kannan Contraction on a metric space and proved a fixed point

[1] Department of Mathematics, The University of Burdwan, Purba Bardhaman-713104, West Bengal, Email: spanja1729@gmail.com

[2] Department of Mathematics, The University of Burdwan, Purba Bardhaman-713104, West Bengal, India, Email: kushal.roy93@gmail.com

[3] Department of Mathematics, The University of Burdwan, Purba Bardhaman-713104, West Bengal, India, Email: mantusaha.bu@gmail.com

theorem of it. Subsequently by re-defining the definition of Interpolative Kannan Contraction, Karapinar et al. [17] had been able to show that the fixed point of Interpolative Kannan contraction mapping may not be unique. Motivated by the background of this literatures, E. Karapinar had been succeed to introduce the notion of interpolative Ćirić-Reich-Rus type mappings for searching of their fixed point theorems on a metric space (see [17]).

In the year 2012 Wardowski [29] introduced a new type of contractive map over a metric space with the help of a certain class of functions $F : (0,\infty) \to \mathbb{R}$, called F-contraction and proved a fixed point theorems over a complete metric space. Subsequently many authors have generalized and extended the result of F-contraction in various ways by the choice of new type of contraction maps and appropriate underlying spaces (for these we may refer [1], [10], [14]).

Inspired by the works of Wardowski's [29] F-contraction, in 2016 Nicolae-Adrian Secelean et al. [27] defined a new contraction mappings called ψF-contraction for a larger class of functions than those of F-contraction by choosing a new class of functions Ψ defined below and proved a fixed point theorem of it over a complete metric space :

$$\Psi := \left\{ \psi : \mathbb{R} \to \mathbb{R} \;\middle|\; \begin{array}{l} \psi \text{ is monotone increasing and} \\ \psi^n(t) \to -\infty \text{ as } n \to \infty \text{ for all } t \in \mathbb{R} \end{array} \right\}$$

where ψ^n denotes the n-th composition of ψ.

Recently in 2019 B. Mohammadi[20] extended the result of Karapinar [17] by using F-contraction, namely *extended interpolative Ćirić-Reich-Rus type F-contraction*.

Influenced by the review of literatures, in this article we introduce the extended version of Reich-Rus type interpolative ψF-contraction map over a b-metric space and prove a fixed point theorem of it. Examples are also given in support of our theorem.

2. **Preliminaries**

Wardowski [29] considered the following class of functions to define F-contraction. Let $F : (0,\infty) \to \mathbb{R}$ be a strictly increasing function satisfying the following conditions:

(F1) For every sequence $\{x_n\} \in (0,\infty)$, $\lim_{n\to\infty} x_n = 0 \iff \lim_{n\to\infty} F(x_n) = -\infty$.

(F2) There exists a constant $k \in (0,1)$ such that $x^k F(x) \to 0$, when $x \to 0^+$.

Let $\mathscr{F}$ denotes the class of all such functions F defined as above. Some examples of such functions could be found in [29]. The following examples are given below for ready references.

Example 2.1. *Let $F : (0,\infty) \to \mathbb{R}$ defined by*

(1) $F(x) = \ln x$.

(2) $F(x) = -\frac{1}{x^m},\ 0 < m < 1$.

(3) $F(x) = x + \ln x$.

(4) $F(x) = \ln(x + x^2)$.

(5) $F(x) = \begin{cases} -\frac{1}{\sqrt{x}} & \text{if } 0 < x \leq \sqrt{3} \\ -\frac{x^{3/2}}{1+x^2} & \text{if } x > \sqrt{3} \end{cases}$

Remark 2.1. *Linear combination of any two members of $\mathscr{F}$ is also a member of $\mathscr{F}$.*

Definition 2.1. [29] *In a metric space* $(\mathcal{M}, d)$ *a mapping* $T : \mathcal{M} \to \mathcal{M}$ *is said to be an* F*-contraction if there exists* $F \in \mathscr{F}$ *and there exists a positive real number* τ *such that*

$$\tau + F(d(T\mu, T\nu)) \le F(d(\mu, \nu)) \tag{1}$$

for all $\mu, \nu \in \mathcal{M}$ *with* $T\mu \ne T\nu$.

In [29] Wardowski proved that, in a complete metric space $(\mathcal{M}, d)$, an F-contraction mapping $T : \mathcal{M} \to \mathcal{M}$ possesses a unique fixed point.

Lemma 2.1. *If* $\psi \in \Psi$ *then* $\psi(t) < t$ *for all* $t \in \mathbb{R}$.

Proof. To the contrary assume that, there exists $t' \in \mathbb{R}$ such that $\psi(t') \ge t'$. Since ψ is increasing so for every natural number n, successively we have $\psi^n(t') \ge \psi^{n-1}(t') \ge \psi^{n-2}(t') \ge \cdots \ge \psi^2(t') \ge \psi(t') \ge t'$, which contradicts that $\psi^n(t) \to -\infty$ as $n \to \infty$. Hence the proof is done. □

Definition 2.2. [27] *In a metric space* $(\mathcal{M}, d)$*, a mapping* $T : \mathcal{M} \to \mathcal{M}$ *is said to be* ψF*-contraction if there exists* $F \in \mathscr{F}$ *and* $\psi \in \Psi$ *such that*

$$F(d(T\mu, T\nu)) \le \psi(F(d(\mu, \nu))) \quad \text{for all } \mu, \nu \in \mathcal{M} \text{ with } T\mu \ne T\nu. \tag{2}$$

A ψF-contractive self-mapping T over a complete metric space possesses a fixed point in it, and fixed point need not necessarily be unique.

Definition 2.3. [17] *In a metric space* $(\mathcal{M}, d)$ *a mapping* $T : \mathcal{M} \to \mathcal{M}$ *is said to be interpolative Kannan type mapping if*

$$d(T\mu, T\nu) \le \kappa (d(\mu, T\mu))^{\alpha} (d(\nu, T\nu))^{1-\alpha} \tag{3}$$

for all $\mu, \nu \in \mathcal{M} \setminus \mathrm{Fix}(T)$ *and for some* $\kappa \in [0, 1)$*,* $\alpha \in (0, 1)$.

In a complete metric space $(\mathcal{M}, d)$, every interpolative Kannan type mapping has a fixed point.

Definition 2.4. [17] *In a metric space* $(\mathcal{M}, d)$ *a mapping* $T : \mathcal{M} \to \mathcal{M}$ *is said to be interpolative Ćirić-Reich-Rus type contraction if there exists* $\kappa \in [0, 1)$ *and* $\alpha, \beta \in (0, 1)$ *with* $\alpha + \beta < 1$ *such that*

$$d(T\mu, T\nu) \le \kappa [d(\mu, \nu)]^{\alpha} [d(\mu, T\mu)]^{\beta} [d(\nu, T\nu)]^{1-\alpha-\beta} \tag{4}$$

for all $\mu, \nu \in \mathcal{M} \setminus \mathrm{Fix}(T)$.

In a complete metric space $(\mathcal{M}, d)$, an interpolative Ćirić-Reich-Rus type contraction admits a fixed point.

Definition 2.5. [20] *In a metric space* $(\mathcal{M}, d)$*, a mapping* $T : \mathcal{M} \to \mathcal{M}$ *is said to be an extended interpolative Ćirić-Reich-Rus type* F*-contraction if there exists* $F \in \mathscr{F}$ *such that for all* $\mu, \nu \in \mathcal{M} \setminus \mathrm{Fix}(T)$ *with* $T\mu \ne T\nu$,

$$\tau + F(d(T\mu, T\nu)) \le \alpha F(d(\mu, \nu)) + \beta F(d(\mu, T\mu)) + (1 - \alpha - \beta) F(d(\nu, T\nu)) \tag{5}$$

for some $\alpha, \beta \in (0, 1)$ *with* $\alpha + \beta < 1$ *and for some positive real* τ.

Theorem 2.1. [20] *In a complete metric space* $(\mathcal{M}, d)$*, an extended interpolative Ćirić-Reich-Rus type* F*-contraction* $T : X \to X$ *assumes a fixed point.*

In 1998, Czerwik initiated the concept of b-metric spaces (see [12]) which is defined as follows. After that several researchers took an attempt in the investigation of the existence of fixed points for single and multivalued mappings and to prove several fixed points theorems on such b-metric spaces. Some of them are referred in the literatures of fixed point theory (see [2], [3], [21], [24], [28]).

Definition 2.6. [12] *Let* $\mathcal{M}$ *be a non-empty set and* $s \geq 1$ *be a real number. Then the mapping* $d : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ *is said to be a* b*-metric on* $\mathcal{M}$ *if the followings hold:*
(B1) $\mu = \nu \iff d(\mu, \nu) = 0$ *for all* $\mu, \nu \in \mathcal{M}$.
(B2) $d(\mu, \nu) = d(\nu, \mu)$ *for all* $\mu, \nu \in \mathcal{M}$.
(B3) $d(\mu, \varrho) \leq s[d(\mu, \nu) + d(\nu, \varrho)]$ *for all* $\mu, \nu, \varrho \in \mathcal{M}$.

We then say that $(\mathcal{M}, d, s)$ is a b-metric space. It is clear that the class of b-metric spaces is larger than the class of metric spaces.

There are various examples of b-metrics which could be found in [7], [13], [22]. The following are some examples of b-metric spaces.

Example 2.2. (1) *Consider the space* $\mathbb{Z}^+$ *of all positive integers. Define* $d : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}$ *by*

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{n} & \text{if } 10^{n-1} \mid (x - y) \quad \text{but } 10^n \nmid (x - y) \quad \text{for } n > 1. \\ \frac{1}{8} & \text{elsewhere} \end{cases}$$

Then $(\mathbb{Z}^+, d)$ *forms a* b*-metric with* $s = 2$.

(2) *Consider that space* $\mathcal{M} := \{p, q, r, s\}$*. Define* $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ *by*

$$d(x, y) = \begin{cases} 0 & \text{when } x = y \\ 5 & \text{when } (x, y) \in \{(p, q), (q, p)\} \\ 1 & \text{when } (x, y) \in \{(q, r), (r, q)\} \\ \frac{1}{4} & \text{when } (x, y) \in \{(p, r), (r, p)\} \\ 3 & \text{when } (x, y) \in \{(p, s), (s, p)\} \\ 2 & \text{when } (x, y) \in \{(q, s), (s, q), (r, s), (s, r)\} \end{cases}$$

Then $(\mathcal{M}, d, 4)$ *forms a* b*-metric space.*

(3) [22] *For any metric* d *on a set* $\mathcal{M}$*, the function* $d'(\mu, \nu) = d(\mu, \nu)^a$*, where* $a (> 1)$ *is real number forms a* b*-metric on* $\mathcal{M}$ *with* $s = 2^{a-1}$.

(4) [7] *The sequence space* $l_p := \left\{ (x_n) \subset \mathbb{R} \;\middle|\; \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$ $(0 < p < 1)$ *with the function* $d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p}$*, where* $x = (x_n)$ *and* $y = (y_n)$ *forms a* b*-metric with*

$s = 2^{1/p}$.

(5) [7] *The space* $L_p[0,1] := \left\{ x(t) \;\middle|\; \int_0^1 |x(t)|^p < \infty \right\}$ $(0 < p < 1)$ *with the function* $d(x,y) = \left(\int_0^1 |x(t) - y(t)|^p \, dt \right)^{1/p}$, *where* $x, y \in L_p[0,1]$ *forms a b-metric with* $s = 2^{1/p}$.

(6) [13] *Consider the space* $\mathcal{M} = [0,2]$ *with the function* $d : \mathcal{M} \times \mathcal{M} \to [0,\infty)$ *defined by*

$$d(\mu,\nu) = \begin{cases} (\mu - \nu)^2 & \text{when } \mu,\nu \in [0,1] \\ \left| \frac{1}{\mu^2} - \frac{1}{\nu^2} \right| & \text{when } \mu,\nu \in [1,2] \\ |\mu - \nu| & \text{elsewhere} \end{cases}$$

Then $(\mathcal{M}, d, 2)$ *forms a b-metric space.*

3. **Main Result**

First we define extended interpolative Reich-Rus type ψF-contraction.

Definition 3.1. *In a b-metric space* $(\mathcal{M}, d, s)$ *with* $s \geq 1$, *we say that a mapping* $T : \mathcal{M} \to \mathcal{M}$ *is an extended interpolative Reich-Rus type* ψF*-contraction if there exist* $F \in \mathscr{F}$ *and* $\psi \in \Psi$ *such that for all* $\mu, \nu \in \mathcal{M} \setminus Fix(T)$ *with* $T\mu \neq T\nu$,

$$F(sd(T\mu, T\nu)) \leq a\psi(F(sd(\mu,\nu))) + b\psi(F(sd(\mu, T\mu))) + c\psi(F(sd(\nu, T\nu))) \tag{6}$$

for some constants $a, b, c \in [0,1]$ *with* $0 < a + b + c \leq 1$.

Theorem 3.1. *Let* $T : \mathcal{M} \to \mathcal{M}$ *be an extended interpolative Reich-Rus type* ψF*-contraction over a complete b-metric space* $(\mathcal{M}, d, s)$. *Let* $\omega_0 \in \mathcal{M} \setminus Fix(T)$ *be such that the series* $\sum_n |\psi^n(F(s\beta_0))|^{-1/k}$ *is convergent, where the constant* $k \in (0,1)$ *comes from* (F2) *and* $\beta_0 = d(\omega_0, T\omega_0)$. *Then* T *admits a fixed point in* $\mathcal{M}$.

Proof. Starting from $\omega_0 \in \mathcal{M} \setminus Fix(T)$, generate the sequence $\{\omega_n\}$ given by $\omega_n = T\omega_{n-1}$ for all $n = 1, 2, \cdots$.
If there exists $m \in \mathbb{N}$ such that $\omega_m = \omega_{m+1}$ then ω_m is a fixed point and then we have done. So suppose $\omega_m \neq \omega_{m+1}$ for all $m \in \mathbb{N}$. Then using 6 we can have,

$$\begin{aligned} F(sd(\omega_{n+1}, \omega_n)) &= F(sd(T\omega_n, T\omega_{n-1})) \\ &\leq a\psi(F(sd(\omega_n, \omega_{n-1}))) + b\psi(F(sd(\omega_n, \omega_{n+1}))) + c\psi(F(sd(\omega_{n-1}, \omega_n))) \end{aligned} \tag{7}$$

Now we claim that $d(\omega_n, \omega_{n+1}) \leq d(\omega_{n-1}, \omega_n)$. If not, then $d(\omega_{n-1}, \omega_n) < d(\omega_n, \omega_{n+1})$. As ψ and F both are increasing so we can write $\psi(F(sd(\omega_{n-1}, \omega_n))) \leq \psi(F(sd(\omega_n, \omega_{n+1})))$. Therefore from (7) we have,

$$\begin{aligned} F(sd(\omega_{n+1}, \omega_n))) &\leq (a + b + c)\psi(F(sd(\omega_{n+1}, \omega_n))) \\ &\leq \psi(F(sd(\omega_{n+1}, \omega_n)))\ , \text{ as } a + b + c \leq 1. \end{aligned}$$

− a contradiction, since $\psi(t) < t$ for all $t \in \mathbb{R}$. Thus our claim stands.
Now by (7) we see that,

$$\begin{aligned} F(sd(\omega_{n+1}, \omega_n)) &\leq (a+b+c)\psi(F(sd(\omega_n, \omega_{n-1}))) \\ &\leq \psi(F(sd(\omega_n, \omega_{n-1}))) \\ &\leq \psi^2(F(sd(\omega_{n-1}, \omega_{n-2}))) \\ &\leq \psi^3(F(sd(\omega_{n-2}, \omega_{n-3}))) \\ &\vdots \\ &\leq \psi^n(F(sd(\omega_1, \omega_0))) \qquad (8) \\ &\longrightarrow -\infty \ \text{ as } n \to \infty. \end{aligned}$$

Therefore $\lim_{n\to\infty} F(sd(\omega_{n+1}, \omega_n)) = -\infty$, which implies that $\lim_{n\to\infty} sd(\omega_{n+1}, \omega_n) = 0$. Define, $\beta_n := d(\omega_n, \omega_{n+1})$. *i.e.*, $s\beta_n \to 0$ as $n \to \infty$. Then there exists $k \in (0, 1)$ such that$(s\beta_n)^k F(s\beta_n) \to 0$ as $n \to \infty$. Again from (8), $F(s\beta_n) \leq \psi^n(F(s\beta_0))$. As $\psi(t) < t$ for all $t \in \mathbb{R}$ we have the relation

$$(s\beta_n)^k F(s\beta_n) \leq (s\beta_n)^k \psi^n((F(s\beta_0)) \leq (s\beta_n)^k (F(s\beta_0)).$$

Thus by Sandwitch Theorem, $\lim_{n\to\infty} \beta_n^k \psi^n(F(s\beta_0)) = 0$. So corresponding to $\epsilon = 1$, there exists $n_0 \in \mathbb{N}$ such that $\beta_n < |\psi^n(F(s\beta_0))|^{-1/k}$, whenever $n \geq n_0$.
Hence,

$$\begin{aligned} d(\omega_n, \omega_{n+p}) &\leq s\left[d(\omega_n, \omega_{n+1}) + d(\omega_{n+1}, \omega_{n+p})\right] \\ &\leq sd(\omega_n, \omega_{n+1}) + s^2\left[d(\omega_{n+1}, \omega_{n+2}) + d(\omega_{n+2}, \omega_{n+p})\right] \\ &\vdots \\ &\leq sd(\omega_n, \omega_{n+1}) + s^2 d(\omega_{n+1}, \omega_{n+2}) + \cdots + s^{p-1} d(\omega_{n+p-2}, \omega_{n+p-1}) \\ &\qquad + s^{p-1} d(\omega_{n+p-1}, \omega_{n+p}) \\ &< s\beta_n + s^2\beta_{n+1} + \cdots + s^p \beta_{n+p-1} \\ &< s^p \sum_{r=n}^{n+p-1} \beta_r < s^p \sum_{r=n}^{n+p-1} |\psi^r(F(s\beta_0))|^{-1/k} \end{aligned}$$

Since the series $\sum_n |\psi^n(F(s\beta_0))|^{-1/k}$ is convergent, so for every $p = 1, 2, \cdots$ we have $d(\omega_n, \omega_{n+p}) \to 0$ as $n \to \infty$ and consequently $\{\omega_n\}$ is a Cauchy sequence in $\mathcal{M}$ and by completeness of $\mathcal{M}$, $\{\omega_n\}$ is convergent. Let $\lim_{n\to\infty} \omega_n = \omega \in \mathcal{M}$.
Now if $T\omega = \omega$ then there is nothing to prove. So suppose $T\omega \neq \omega$.
Then

$$\begin{aligned} F(sd(\omega_{n+1}, T\omega) &= F(sd(T\omega_n, T\omega) \\ &\leq a\psi(F(sd(\omega_n, \omega))) + b\psi(F(sd(\omega_n, \omega_{n+1}))) + c\psi(F(sd(\omega, T\omega))) \\ &\leq aF(sd(\omega_n, \omega)) + bF(sd(\omega_n, \omega_{n+1})) + cF(sd(\omega, T\omega)) \\ &\longrightarrow -\infty \ , as \ n \to \infty. \end{aligned}$$

follows that $\lim_{n\to\infty} d(\omega_{n+1}, T\omega) = 0$. *i.e.*, $d(\lim_n \omega_{n+1}, T\omega) = 0$, *i.e.*, $d(\omega, T\omega) = 0$, proves that $\omega \in \mathcal{M}$ is a fixed point of T. $\square$

Example 3.1. *Consider the sapce $\mathcal{M} = \{1, 2, 3, 4\}$ and define a function $d : \mathcal{M}\times\mathcal{M} \to [0, \infty)$ by*

$$d(\mu,\nu) = \begin{cases} 0 & \text{when } \mu = \nu \\ 2 & \text{when } (\mu,\nu) \in \{(1,2),(2,1)\} \\ \frac{1}{2} & \text{when } (\mu,\nu) \in \{(2,3),(3,2)\} \\ 1 & \text{when } (\mu,\nu) \in \{(1,3),(3,1)\} \\ \frac{3}{2} & \text{when } (\mu,\nu) \in \{(1,4),(4,1)\} \\ 3 & \text{when } (\mu,\nu) \in \{(2,4),(4,2),(3,4),(4,3)\} \end{cases}$$

Then $(\mathcal{M}, d, s)$ forms a b-metric space with $s = 2$.

Define a map $T : \mathcal{M} \to \mathcal{M}$ by $1 \mapsto 3$, $2 \mapsto 1$, $3 \mapsto 3$ and $4 \mapsto 4$.

Choose $F(x) = x + \ln x$ and $\psi(t) = t - \frac{1}{10}$. Then $F \in \mathscr{F}$ with $k = 2$ in (F2) *and $\psi \in \Psi$ where $\psi^n(t) = t - \frac{n}{10}$. Then T is an extended interpolative Reich-Rus type ψF-contraction* (6) *with $a = b = c = \frac{1}{3}$, for the chosen $F \in \mathscr{F}$ and $\psi \in \Psi$.*

If we take $\omega_0 = 1$ then $\beta_0 = d(\omega_0, T\omega_0) = 1$ and then the series $\sum_n |\psi^n(F(s\beta_0))|^{-1/k}$ becomes convergent. Thus all the conditions of the Theorem 3.1 *are satisfied.*

Clearly it is seen that $\mu = 3$ and $\mu = 4$ both are fixed points of T in $\mathcal{M}$.

Example 3.2. *The space $\mathcal{M} = [0, 4] \cup \{5, 6\}$ equipped with $d(\mu, \nu) = |\mu-\nu|^2$ for all $\mu, \nu \in \mathcal{M}$ forms a b-metric with $s = 2$.*

Define a map $T : \mathcal{M} \to \mathcal{M}$ by

$$T(\mu) = \begin{cases} 5 & , \text{ if } \mu \in [0,4] \cup \{5\} \\ 4 & , \text{ if } \mu = 6 \end{cases}$$

Take $F(x) = \ln x$. Then $F \in \mathscr{F}$ with $k = \frac{1}{2}$ in (F2)*; and take $\psi(t) = t - \frac{1}{2} \in \Psi$. Then T is an extended interpolative Reich-Rus type ψF contraction* (6) *with $a = b = c = \frac{1}{3}$, for the chosen F and ψ.*

Also it can be easily verified that all the conditions of the Theorem 3.1 *are satisfied. Clearly T has a fixed point $5 \in \mathcal{M}$.*

Corollary 3.1. *If we take $a = 1$, $b = 0$, $c = 0$ and $s = 1$ in* (6) *then it reduces to ψF-contraction* (2)*.*

There may be a mapping $T : \mathcal{M} \to \mathcal{M}$ which is an extended interpolative Reich-Rus type ψF-contraction but not necessarily a ψF-contraction. The following example supports our contention.

Example 3.3. *Consider the space $\mathcal{M} = \{1, 2, 4, 5, 6\} \subset \mathbb{R}$ equipped with the usual metric d_u of $\mathbb{R}$. Then X is complete metric space trivially* (*more precisely X is a complete b-metric space with $s = 1$*)*.*

Define a map $T : \mathcal{M} \to \mathcal{M}$ by

$$T(\mu) = \begin{cases} 4 & , \text{ if } \mu = 1 \\ 5 & , \text{ if } \mu = 2 \\ 6 & , \text{ if } \mu \in \{4, 5, 6\} \end{cases}$$

We claim that T is not a ψF-contraction map. If it is so, then by taking $\mu = 1, \nu = 2$ and using Lemma 2.1 we have,

$$F(1) = F(d_u(T\mu, T\nu)) \leq \psi(F(d_u(\mu, \nu))) < F(d_u(\mu, \nu)) = F(1), \text{ a contradiction.}$$

Now we will show that T is an extended interpolative Reich-Rus type ψF-contraction. To verify this we choose $F(x) = \ln x$, $\psi(t) = t - 0.1$ and $a = 0.1$, $b = 0.8$, $c = 0.1$.

Case 1: *When $\mu = 1$ and $\nu = 2$ then,*

$$\begin{aligned} 0 = F(1) = F(d_u(T\mu, T\nu)) &\leq a\psi(F(d_u(\mu, \nu))) + b\psi(F(d_u(\mu, T\mu))) + c\psi(F(d_u(\nu, T\nu))) \\ &= a\psi(F(1)) + b\psi(F(3)) + c\psi(F(3)) \\ &= 0.1(0 - 0.1) + 0.8(\ln 3 - 0.1) + 0.1(\ln 3 - 0.1) \\ &\approx 1.59 \end{aligned}$$

Case 2: *When $\mu = 1$ and $\nu = 4$ then,*

$$\begin{aligned} 0.69 \approx F(2) = F(d_u(T\mu, T\nu)) &\leq a\psi(F(d_u(\mu, \nu))) + b\psi(F(d_u(\mu, T\mu))) + c\psi(F(d_u(\nu, T\nu))) \\ &= a\psi(F(3)) + b\psi(F(3)) + c\psi(F(2)) \\ &= 0.1(\ln 3 - 0.1) + 0.8(\ln 3 - 0.1) + 0.1(\ln 2 - 0.1) \\ &\approx 0.96 \end{aligned}$$

Case 3: *When $\mu = 1$ and $\nu = 5$ then,*

$$\begin{aligned} 0.69 \approx F(2) = F(d_u(T\mu, T\nu)) &\leq a\psi(F(d_u(\mu, \nu))) + b\psi(F(d_u(\mu, T\mu))) + c\psi(F(d_u(\nu, T\nu))) \\ &= a\psi(F(4)) + b\psi(F(3)) + c\psi(F(1)) \\ &= 0.1(\ln 4 - 0.1) + 0.8(\ln 3 - 0.1) + 0.1(0 - 0.1) \\ &\approx 0.92 \end{aligned}$$

Case 4: *When $\mu = 2$ and $\nu = 4$ then,*

$$\begin{aligned} 0 = F(1) = F(d_u(T\mu, T\nu)) &\leq a\psi(F(d_u(\mu, \nu))) + b\psi(F(d_u(\mu, T\mu))) + c\psi(F(d_u(\nu, T\nu))) \\ &= a\psi(F(2)) + b\psi(F(3)) + c\psi(F(2)) \\ &= 0.1(\ln 2 - 0.1) + 0.8(\ln 3 - 0.1) + 0.1(\ln 2 - 0.1) \\ &\approx 0.92 \end{aligned}$$

Case *5: When* $\mu = 2$ *and* $\nu = 5$ *then,*

$$\begin{aligned} 0 = F(1) = F(d_u(T\mu, T\nu)) &\leq a\psi(F(d_u(\mu,\nu))) + b\psi(F(d_u(\mu,T\mu))) + c\psi(F(d_u(\nu,T\nu))) \\ &= a\psi(F(3)) + b\psi(F(3)) + c\psi(F(1)) \\ &= 0.1(\ln 3 - 0.1) + 0.8(\ln 3 - 0.1) + 0.1(0 - 0.1) \\ &\approx 0.89 \end{aligned}$$

Thus all the cases are verified. Moreover it can be shown that T satisfies all the conditions of the Theorem 3.1*. Also it is clear that $\mu = 6 \in \mathcal{M}$ is a fixed point of T.*

Corollary 3.2. *If we consider in particular, $\psi(t) = t - k$ for some constant $k > 0$ and $s = 1$ then* (6) *turns into*

$$F(d(T\mu, T\nu)) \leq aF(d(\mu,\nu)) + bF(d(\mu,T\mu)) + cF(d(\nu,T\nu)) - (a+b+c)k.$$

which is nothing but extended interpolative Ćirić-Reich-Rus type F-contraction (5) *with $\tau = (a+b+c)k > 0$.*

Corollary 3.3. *Assuming the constants $a > 0$, $b > 0$ such that $a+b+c = 1$ and $\psi(t) = t - p$ for some constant $p > 0$, we consider $F(x) = \ln x$ and $s = 1$ in* (6)*. Then we get,*

$$\ln d(T\mu, T\nu) \leq a \ln d(\mu,\nu) + b \ln d(\mu,T\mu) + c \ln d(\nu,T\nu) - p,$$

from which it follows that T is an interpolative Ćirić-Reich-Rus type contraction (4) *with $\kappa = e^{-p} \in [0,1)$.*

4. An Application to Dynamical Programming

In 1957, Bellman [5] first introduced the existence and successive approximations of solutions for the various functional equations arising in dynamical programming, one of them is as follows:

$$f(t) = \sup_{t' \in \mathcal{D}} \{p(t,t') + G(t,t',f(q(t,t')))\} \tag{9}$$

where X and Y are Banach spaces over the field $\mathbb{R}$. $\mathcal{S} \subset X$ denotes the state space (i.e., the set of initial state, action and transition of the process) and $\mathcal{D} \subset Y$ denotes the decision space (i.e., the set of all possible actions of the process) and t, t' denotes the state and decision vectors respectively.
Here, $f(t)$ denotes the optimal return function with initial state t and $p : \mathcal{S} \times \mathcal{D} \to \mathbb{R}$, $q : \mathcal{S} \times \mathcal{D} \to \mathcal{S}$, $G : \mathcal{S} \times \mathcal{D} \times \mathbb{R} \to \mathbb{R}$. Thereafter several authors analyses the properties of solution of several functional equations arising in dynamical programming and solve them by using various fixed point theorems. For these we may refer to [6], [8], [18], [19].

Let $B_d(\mathcal{S})$:= set of all real-valued bounded functions on $\mathcal{S}$. We define a norm on $B_d(\mathcal{S})$ by $\|\theta\| = \sup_{t\in\mathcal{S}}|\theta(t)|$ for all $\theta \in B_d(\mathcal{S})$. Then $(B_d(\mathcal{S}), \|.\|)$ forms a Banach space equipped with the metric defined by $d(\theta,\vartheta) := \sup_{t\in\mathcal{S}}|\theta(t) - \vartheta(t)|$ for all $\theta, \vartheta \in B_d(\mathcal{S})$.

Now we are going to discuss the existence of the solution for the functional equation (9) by using Extended interpolative Reich-Rus type ψF-contraction. For this first we define an operator $\Upsilon : B_d(\mathcal{S}) \to B_d(\mathcal{S})$ by

$$(\Upsilon f)(t) = \sup_{t' \in \mathcal{D}} \{p(t,t') + G(t,t',f(q(t,t'))\}. \tag{10}$$

for all $f \in B_d(\mathcal{S})$. Clearly if the functions p and G are bounded then Υ becomes well-defined.

Theorem 4.1. *Let $\Upsilon : B_d(\mathcal{S}) \to B_d(\mathcal{S})$ be an operator defined by* (10) *and suppose the following conditions hold.*
(A1) $p : \mathcal{S} \times \mathcal{D} \to \mathbb{R}$ and $G : \mathcal{S} \times \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ are continuous and bounded.
(A2) For all $\theta, \vartheta \in B_d(\mathcal{S}) \setminus Fix(\Upsilon)$, G satisfies

$$|G(t,t',\theta(t)) - G(t,t',\vartheta(t))| \leq \frac{1}{e}\sqrt[3]{d(\theta,\vartheta).d(\theta,\Upsilon\theta).d(\vartheta,\Upsilon\vartheta)} \tag{11}$$

for all $(t,t',\theta(t)),(t,t',\vartheta(t)) \in \mathcal{S} \times \mathcal{D} \times \mathbb{R}$. Then the functional equation (9) *has a bounded solution.*

Proof. Let $\epsilon > 0$ be arbitrary real number and let $\theta, \vartheta \in B_d(\mathcal{S}) \setminus Fix(\Upsilon)$. Then there exists $t_1', t_2' \in \mathcal{D}$ such that for all $t \in \mathcal{S}$,

$$p(t,t_1') + G(t,t_1',\theta(q(t,t_1'))) > \Upsilon\theta(t) - \epsilon\ , \tag{12}$$

$$p(t,t_2') + G(t,t_2',\vartheta(q(t,t_2'))) > \Upsilon\vartheta(t) - \epsilon\ , \tag{13}$$

$$\Upsilon\theta(t) \geq p(t,t_2') + G(t,t_2',\theta(q(t,t_2')))\ , \tag{14}$$

$$\Upsilon\vartheta(t) \geq p(t,t_1') + G(t,t_1',\vartheta(q(t,t_1'))) \tag{15}$$

Now equations (12) and (15) yield the following

$$\begin{aligned}\Upsilon\theta(t) - \Upsilon\vartheta(t) &< G(t,t_1',\theta(q(t,t_1'))) - G(t,t_1',\vartheta(q(t,t_1'))) + \epsilon \\ &\leq |G(t,t_1',\theta(q(t,t_1'))) - G(t,t_1',\vartheta(q(t,t_1')))| + \epsilon \\ &\leq \frac{1}{e}\sqrt[3]{d(\theta,\vartheta).d(\theta,\Upsilon\theta).d(\vartheta,\Upsilon\vartheta)} + \epsilon\ , \text{ for all } t \in \mathcal{S} \end{aligned} \tag{16}$$

Similarly equations (13) and (14) yield

$$\Upsilon\vartheta(t) - \Upsilon\theta(t) < \frac{1}{e}\sqrt[3]{d(\theta,\vartheta).d(\theta,\Upsilon\theta).d(\vartheta,\Upsilon\vartheta)} + \epsilon\ , \text{ for all } t \in \mathcal{S} \tag{17}$$

Now combining (16) and (17) we can write

$$|\Upsilon\theta(t) - \Upsilon\vartheta(t)| < \frac{1}{e}\sqrt[3]{d(\theta,\vartheta).d(\theta,\Upsilon\theta).d(\vartheta,\Upsilon\vartheta)} + \epsilon\ , \text{ for all } t \in \mathcal{S}. \tag{18}$$

Since $\epsilon > 0$ is arbitrary, we have $|\Upsilon\theta(t) - \Upsilon\vartheta(t)| \leq \frac{1}{e}\sqrt[3]{d(\theta,\vartheta).d(\theta,\Upsilon\theta).d(\vartheta,\Upsilon\vartheta)}$, for all $t \in \mathcal{S}$ and thus $d(\Upsilon\theta,\Upsilon\vartheta) \leq \frac{1}{e}\sqrt[3]{d(\theta,\vartheta).d(\theta,\Upsilon\theta).d(\vartheta,\Upsilon\vartheta)}$. Taking logarithm both sides,

$$\begin{aligned}\ln d(\Upsilon\theta,\Upsilon\vartheta) &\leq -1 + \frac{1}{3}\ln d(\theta,\vartheta) + \frac{1}{3}\ln d(\theta,\Upsilon\theta) + \frac{1}{3}\ln d(\vartheta,\Upsilon\vartheta) \\ &= \frac{1}{3}[\ln d(\theta,\vartheta) - 1] + \frac{1}{3}[\ln d(\theta,\Upsilon\theta) - 1] + \frac{1}{3}[\ln d(\vartheta,\Upsilon\vartheta) - 1].\end{aligned}$$

It is observed that the above inequality is nothing but extended interpolative Reich-Rus type ψF-contraction (6) for the operator Υ, taking $F(x) = \ln x \in \mathscr{F}$, $\psi(x) = x - 1 \in \Psi$, $a = b = c = \frac{1}{3}$ and $s = 1$.

Again we have $\psi^n(x) = x - n$ and for any $\theta_0 \in B_d(\mathcal{S}) \setminus Fix(\Upsilon)$, $\beta_0 = d(\theta_0, \Upsilon\theta_0) = \sup_{x\in\mathcal{S}} |\theta_0(x) - \Upsilon\theta_0(x)| \in \mathbb{R}^+$. Then the series $\sum_n |\psi^n(F(\beta_0)|^{-1/k}$ becomes convergent whatever the value of $k \in (0,1)$. Hence by Theorem 3.1, Υ has a fixed point in $B_d(\mathcal{S})$ and consequently the functional equation (9) has a bounded solution.

□

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