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MAZUR-ORLICZ THEOREM IN CONCRETE SPACES AND INVERSE PROBLEMS RELATED TO THE MOMENT PROBLEM

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In the first part of this work, we derive some new applications of a version of Mazur-Orlicz theorem, in concrete spaces of absolutely integrable functions and respectively continuous functions of several real variables. The second part is devoted to inverse problems related to the Markov moment problem. A geometric approach of approximating the solutions of a system with infinitely many equations involving transcendent functions, with infinitely many unknowns, is briefly discussed.

Keywords: Mazur-Orlicz theorem, L^p spaces, self – adjoint operators, Markov moment problem, inverse problems
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1. Introduction

The version of Mazur - Orlicz theorem that we have in mind in this work gives a necessary and sufficient condition for the existence of a linear positive operator F from an order vector space X into an order complete vector lattice Y, such that $F(x_j) \geq y_j, j \in J, F(x) \leq P(x), x \in X$, where $\{x_j\}_{j\in J} \subset X,\ \{y_j\}_{j\in J} \subset Y$ are given families, $P: X \to Y$ being a sublinear operator [1]. The relation $F(x) \leq P(x),\ x \in X$ usually controls the norm of the solution F. Recent results on this subject have been published in [2] and have been submitted in [4]. The first aim of this work is to prove some new application of Mazur – Orlicz theorem to concrete spaces X, namely to $X = L^p, 1 \leq p < \infty$. The second purpose of this work is to solve an inverse problem related to a Markov moment problem (see the Abstract). From this point of view, one continues the study started in [3], [13]. An existence result for the solution of a Markov moment problem [1] is applied. For similar existence problems based on Hahn – Banach theorem and its generalizations see [2] - [9], [13]. For operator valued moment problems see [9] – [12]. For the construction of some solutions see [9], [13], [3]. The purpose of the second part of this work is to approximate the solution of a system with infinitely many equations involving transcendent functions, with infinitely many unknowns, starting from the moments of a solution of a Markov moment problem. Our solution is not unique. This is

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another way of solving similar problems to those treated in literature by some other methods [14]. Recall that another important problem in the theory of moments is the uniqueness of the solution [15] - [18]. The background of this work is contained in some chapters from [19] – [22]. The rest of the paper is organized as follows. Section 2 is devoted to some applications of Mazur – Orlicz theorem. In Section 3, inverse problems related to the Markov moment problem are discussed. The conclusions are mentioned in Section 4.

2. Applications of Mazur – Orlicz theorem

We start by recalling the general abstract form of Mazur – Orlicz theorem, in the order vector spaces setting.

Theorem 2.1. (Theorem 5 [1]). *Let* $\boldsymbol{X}$ *be an ordered vector space,* $\boldsymbol{Y}$ *an order complete vector lattice,* $\{\boldsymbol{x_j}\}_{\boldsymbol{j\in J}}, \{\boldsymbol{y_j}\}_{\boldsymbol{j\in J}}$ *arbitrary families in* $\boldsymbol{X}$*, respectively in* $\boldsymbol{Y}$ *and* $\boldsymbol{P}: \boldsymbol{X} \to \boldsymbol{Y}$ *a sublinear operator. The following statements are equivalent*

(a) $\exists F \in L(X,Y)$ *such that* $F(x_j) \geq y_j, \forall j \in J, F(x) \geq 0\ \forall x \in X_+,$

$$F(x) \leq P(x), \forall x \in X;$$

(b) *for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j\}_{j\in J_0} \subset \mathbb{R}, \lambda_j \geq 0\ \forall j \in J_0$*, we have*

$$\sum_{j\in J_0} \lambda_j\, x_j \leq x \in X \Rightarrow \sum_{j\in J_0} \lambda_j\, y_j \leq P(x).$$

Some new applications of this general result are deduced in the sequel.

Theorem 2.2. *Let* X *be a Banach lattice,* Y *an order complete Banach lattice,* $\{\varphi_j\}_{j\in J} \subset X_+, \{y_j\}_{j\in J} \subset Y$*,* G *a linear positive bounded operator from* X *into* Y*,* α *a positive number. The following statements are equivalent*

(a) *there exists a linear positive bounded operator* $F \in B_+(X,Y)$*, such that*

$$F(\varphi_j) \geq y_j, \forall j \in J, F(x) \leq \alpha G(|x|), \forall x \in X, \|F\| \leq \alpha\|G\|;$$

(b) $y_j \leq \alpha G(\varphi_j), \forall j \in J.$

Proof. (a)⇒(b) is obvious, because of $y_j \leq F(\varphi_j) \leq \alpha G(|\varphi_j|) = \alpha G(\varphi_j), \forall j \in J$. For the converse, we apply Theorem 2.1, (b)⇒(a). Let $J_0 \subset J$ be a finite subset, $\{\lambda_j\}_{j \in J_0} \subset \mathbb{R}_+, x \in X$, such that $\sum_{j \in J_0} \lambda_j \varphi_j \leq x$. Then using (b) and the fact that the scalars λ_j are nonnegative, as well as the positivity of G, we derive

$$\sum_{j \in J_0} \lambda_j y_j \leq \alpha \sum_{j \in J_0} \lambda_j G(\varphi_j) = \alpha G\left(\sum_{j \in J_0} \lambda_j \varphi_j\right) \leq \alpha G(x) \leq \alpha G(|x|) =: P(x).$$

Application of Theorem 2.1 leads to the existence of a linear positive operator F from X into Y such that

$$F(\varphi_j) \geq y_j, \forall j \in J, F(x) \leq \alpha G(|x|), \forall x \in X.$$

From the last relation, also using the fact that the norms on X and Y are solid ($|u| \leq |v| \Rightarrow \|u\| \leq \|v\|$), we deduce

$$|F(x)| \leq \alpha G(|x|) \Rightarrow \|F(x)\| \leq \alpha \|G\| \||x|\| = \alpha \|G\| \|x\|, \forall x \in X.$$

It follows that $\|F\| \leq \alpha \|G\|$. This concludes the proof. □

Corollary 2.1. *Let* M *be a measure space,* μ *a positive measure on* $M, \mu(M) < \infty, X = L_\mu^p(M), 1 \leq p < \infty, g \geq 0$ *an element of* $L_\mu^q(M)$, *where* $q \in (1, \infty]$ *is the conjugate of* p $(1/p + 1/q = 1)$, α *a positive number. Let* $\{\varphi_j\}_{j \in J}, \{y_j\}_{j \in J}$ *be as in Theorem 2.2, where* $Y = \mathbb{R}$. *The following statements are equivalent*

(a) *there exists* $h \in L_\mu^q(M)$, $0 \leq h \leq \alpha g$ *a.e.,* $\int_M h\varphi_j d\mu \geq y_j, \forall j \in J$;

(b) $y_j \leq \alpha \int_M g\varphi_j d\mu, \forall j \in J$.

Proof. One applies Theorem 2.2 for $G(\psi) = \int_M g\, \psi d\mu, \psi \in X, Y = \mathbb{R}$, as well as the representation of linear positive continuous functionals on L^p spaces by means of nonnegative elements from L^q spaces. In order to prove (b)⇒(a), from the preceding results it follows that there exists $h \in \left(L_\mu^q(M)\right)_+$ such that $\int_M h\varphi_j\, d\mu \geq y_j, \forall j \in J$ and

$$\int_M h\psi d\mu \leq \alpha \int_M g\psi d\mu$$

for all nonnegative functions $\psi \in L^p_\mu(M)$. Now we choose $\psi = \chi_B$, where B is an arbitrary measurable subset of M. Then the last relation can be rewritten as

$$\int_B (h - \alpha g)\, d\mu \leq 0$$

for all such subsets B. A straightforward application of Theorem 1.40 [21], leads to $h - \alpha g \leq 0$ a.e. in M. Since (a)⇒(b) is obvious, this concludes the proof. □

Corollary 2.2. *Let consider the measure space* $M = \mathbb{R}^n_+, n \in \{1,2 \dots\}$, *endowed with the measure* $d\mu = exp\left(-\sum_{j=1}^n p_j t_j\right) dt_1 \cdots dt_n, p_j > 0, \forall j \in \{1, \dots, n\}$, α *a positive number. The following statements are equivalent*

(a) *there exists* $h \in L^\infty_\mu(\mathbb{R}^n_+), \int_{\mathbb{R}^n_+} ht^j d\mu \geq y_j, \forall j \in \mathbb{N}^n, 0 \leq h \leq \alpha$ *a.e.;*

(b) $y_j \leq \alpha \frac{J_1! \cdots J_n!}{p_1^{j_1+1} \cdots p_n^{j_n+1}}, \forall j = (j_1, \dots, j_n) \in \mathbb{N}^n$.

Proof. One applies Corollary 2.1 to $p = 1, q = \infty, g = 1$ a.e. The notation t^j is the multi – index notation $t^j = t_1^{j_1} \cdots t_n^{j_n}$. The conclusion follows via Fubini 's theorem and Gamma function properties. □

Theorem 2.3. Let $X = L^p_\mu(M), 1 < p, \mu \geq 0, \mu(M) < \infty, \{\varphi_j\}_{j \in J} \subset X, \{y_j\}_{j \in J} \subset \mathbb{R}, \alpha > 0, \alpha \in \mathbb{R}$, q the conjugate of p. *Consider the following statements*

(a) *there exists* $h \in \left(L^q_\mu(M)\right)_+$ *such that*

$$\int_M h\varphi_j d\mu \geq y_j, \forall j \in J, \int_M h\psi d\mu \leq \alpha \|\psi\|_p \left(\mu(M)\right)^{1/q}, \qquad \forall \psi \in X;$$

(b) *we have* $y_j \leq \alpha \int_M \varphi_j \, d\mu, \forall j \in J$.

Then (b)⇒(a).

Proof. Let $J_0 \subset J$ be a finite subset, $\{\lambda_j\}_{j \in J_0} \subset \mathbb{R}_+$. Hölder inequality and using also (b), lead to the following implications

$$\sum_{j\in J_0}\lambda_j\varphi_j \le \psi \Rightarrow \int_M \left(\sum_{j\in J_0}\lambda_j\varphi_j\right)d\mu \le \int_M \psi d\mu \le \|\psi\|_p\big(\mu(M)\big)^{1/q} \Rightarrow$$

$$\sum_{j\in J_0}\lambda_j y_j \le \alpha\int_M \left(\sum_{j\in J_0}\lambda_j\varphi_j\right)d\mu \le \alpha\,\|\psi\|_p\big(\mu(M)\big)^{1/q} =: P(\psi).$$

Application of Theorem 2.1 and measure theory arguments [21, theorem 6.16, p. 122-124], yield the existence of $h \in L_\mu^q(M)$ such that

$$F(\varphi_j) = \int_M h\varphi_j d\mu \ge y_j\,, \forall j \in J, F(\psi) = \int_M h\psi d\mu \le \alpha\|\psi\|_p\big(\mu(M)\big)^{1/q},\ \psi \in X.$$

Moreover, since $F(\psi) \ge 0, \forall\psi \in X_+$, we have

$$\int_M h\psi d\mu \ge 0, \forall\psi \in X_+.$$

Taking $\psi = \chi_B$, where $B \subset M$ is a measurable set such that $\mu(B) > 0$, one obtains

$$\int_B h d\mu \ge 0$$

for all such subsets B. Application of theorem 1.40 [21] leads to $h \ge 0\ \mu - a.e.$ From the previous relations we also derive that $\|h\|_q \le \alpha\big(\mu(M)\big)^{1/q}$. This concludes the proof. □

The following theorem represents an application of the general result stated in Theorem 2.1 to some other concrete spaces X, Y. Let H be an arbitrary Hilbert space, $n \in \mathbb{N}$, $n \ge 1$, $A_1, \ldots, A_n$ positive commuting self - adjoint operators acting on H, $(B_j)_{j\in\mathbb{N}^n}$ a sequence in Y, where $Y = Y(A_1, \ldots, A_n)$ is defined by

$$Y_1 := \{U \in \mathcal{A}(H); UA_j = A_jU, j = 1, \ldots, n\}, Y := \{V \in Y_1; UV = VU, \forall U \in Y_1\}.$$

Here $\mathcal{A}(H)$ is the real vector space of all self – adjoint operators. One can prove that Y is an order complete Banach lattice with respect to the usual structures induced by those defined on the real space of self – adjoint operators (see [19, p. 303 - 305]), and a commutative real Banach algebra. Notice that the properties of $Y = Y(A_1, \dots, A_n)$, where $A_1, \dots, A_n$ are as mentioned above can be proved in a similar way to those of a $Y(A)$, where A is a self – adjoint operator. Actually, one repeats the proofs from [19, p. 303 – 305], but for several commuting self – adjoint operators. Then Y endowed with the usual order relation on self - adjoint operators is an order - complete vector lattice and a commutative real Banach algebra [19]. Let also be B the C^*-algebra generated by $A = (A_1, A_2, \dots, A_n)$, that is B is the closure in $B(H)$ of expressions of the form

$$P(A_1, A_2, \dots, A_n) = \sum_{\substack{j\in J \\ J\subset\mathbb{N}^n, J\ finite}} a_j A_1^{j_1} A_2^{j_2} \cdots A_n^{j_n},\ a_j \in \mathbb{C}, j = (j_1, \dots, j_n).$$

We may uniquely construct the joint spectral measure E_A of the commuting $A = (A_1, A_2, \dots, A_n)$ in B. As it is known, the joint spectral measure E_A is concentrated on the joint spectrum $\Sigma_A \coloneqq \{\gamma(A_1), \dots, \gamma(A_n);\ \gamma \in \Gamma\} \subset \sigma_B(A_1)x \cdots \times \sigma_B(A_n) \subset \mathbb{R}^n$

$$\Gamma \coloneqq \{\gamma: B \to \mathbb{C}; \gamma \text{ is a } character\},$$

Because for any set $\sigma \in Bor(\Sigma_A)$, we have $E_A(\sigma)\mathrm{A_i} = \mathrm{A}E_{A_\mathrm{i}}$, $i = 1,2, \dots, n$, it results that $E_A(\sigma) \subset Y$. Consequently, we have

$$E_A: Bor(\Sigma_A) \to \mathcal{A}(H).$$

The spectral measure $E_A = E_{(A_1,\dots,A_n)}: Bor(\Sigma_A) \to \mathcal{A}(H)$ is such that for any polynomial $p = p(t_1, t_2, \dots, t_n), (t_1, t_2, \dots, t_n) \in \Sigma_A$ of n real variables, we have

$$\int_{\Sigma_A} p(t_1, \dots, t_n)\, dE_{(A_1,\dots,A_n)} = p(A_1, \dots, A_n)$$

Let denote by $\varphi_j, j \in \mathbb{N}^n$ the basic polynomials $\varphi_j(t_1, \dots, t_n) = t_1^{j_1} \cdots t_n^{j_n}, j = (j_1, \dots, j_n) \in \mathbb{N}^n, t = (t_1, \dots, t_n) \in \Sigma_A,\ X := C(\Sigma_A)$.

Theorem 2.4. *The following statements are equivalent*

(a) *there exists a linear bounded positive operator $F \in B_+(X, Y)$ such that*

$$F(\varphi_j) \geq B_j, j \in \mathbb{N}^n, F(\varphi) \leq \int_{\Sigma_A} |\varphi|\, dE_{(A_1,\dots,A_n)}, \forall\varphi \in X, \|F\| \leq 1;$$

(b) $B_j \le A^j := A_1^{j_1} \cdots A_n^{j_n}, \forall j = (j_1, \dots, j_n) \in \mathbb{N}^n$.

Proof. The implication (a)⇒(b) is obvious:

$$B_j \le F(\varphi_j) \le \int_{\Sigma_A} |\varphi_j| dE_{(A_1,\dots,A_n)} = \int_{\Sigma_A} \varphi_j dE_{(A_1,\dots,A_n)} = A_1^{j_1} \cdots A_n^{j_n},$$

$j \in \mathbb{N}^n$ (we have used the positivity of the operators A_k which leads to $|\varphi_j| = \varphi_j$ on Σ_A). For the converse, one applies Theorem 2.1, (b)⇒(a), where $\mathbb{N}^n$ stands for J, φ_j stands for x_j and B_j stands for $y_j, \forall j \in \mathbb{N}^n$. Let J_0 and $\{\lambda_j\}_{j\in J_0}$ be as mentioned at point (b) of Theorem 2.1. The following implications hold:

$$\sum_{j\in J_0} \lambda_j \varphi_j \le \varphi \in X \Rightarrow \sum_{j\in J_0} \lambda_j \int_{\Sigma_A} \varphi_j dE_{(A_1,\dots,A_n)} = \sum_{j\in J_0} \lambda_j A_1^{j_1} \cdots A_n^{j_n}$$

$$\le \int_{\Sigma_A} \varphi dE_{(A_1,\dots,A_n)} \le \int_{\Sigma_A} |\varphi| \, dE_{(A_1,\dots,A_n)} =: P(\varphi).$$

The positivity of the spectral measure $dE_{(A_1,\dots,A_n)}$ has been used. On the other hand, the hypothesis (b), the fact that the scalars λ_j are nonnegative and the preceding evaluation yield

$$\lambda_j B_j \le \lambda_j A^j \; \forall j \Rightarrow \sum_{j\in J_0} \lambda_j B_j \le \sum_{j\in J_0} \lambda_j A^j = \sum_{j\in J_0} \lambda_j A_1^{j_1} \cdots A_n^{j_n} \le P(\varphi),$$

where $P(\varphi)$ was defined above. Thus the implication at (b) Theorem 2.1 is accomplished. Application of the latter theorem leads to the existence of a "feasible solution" F having the property mentioned at point (a) of the present theorem. The last property is a consequence of the preceding one, using the fact that the norm on Y is solid. This concludes the proof. □

Remark. If in Theorem 2.4. one additionally assumes that $\|A_k\| < 1, k = 1,2,\dots,n$, then for any self - adjoint operators satisfying (b) one has

$$\sum_{j\in\mathbb{N}^n} B_j \le \prod_{k=1}^{n} (I - A_k)^{-1}.$$

3. An inverse problem related to a Markov moment problem

Let $A=(0,1)^n$, $d\nu=(-\ln t_1)dt_1\cdots(-\ln t_n)dt_n$. Assume that there exists a $h\in L_\nu^\infty(A)$, $0\le h\le 1$ *a.e.*, such that

$$m_j=\int_K t_1^{j_1}(-\ln(t_1)))\cdots t_n^{j_n}(-\ln(t_n))h(t_1,\dots,t_n)dt_1\cdots dt_n,$$
$$j=(j_1,\dots,j_n),\ j_k\in N,\ k=1,\dots,n.$$

Denote $\varphi_j(t_1,\dots,t_n)=\cdot t_1^{j1}\cdots t_n^{jn}$, $j_k\in\{0,1,2\dots\}, k=1,\dots,n,(t_1,\dots,t_n)\in A.$ Consider the system of equations

$$m_j=\int_A\varphi_j\cdot h\,d\nu\approx\int_A\varphi_j\cdot\tilde{h}\,d\nu\approx$$
$$\sum_{p,q\le M}c_{p,q}\left(\sum_{m\in\mathrm{N}}\prod_{k=1}^{n}\left(\frac{x_{k,m,p,q}^{j_k+1}\ln(x_{k,m,p,q})-y_{k,m,p,q}^{j_k+1}\ln(y_{k,m,p,q})}{j_k+1}+\frac{y_{k,m,p,q}^{j_k+1}-x_{k,m,p,q}^{j_k+1}}{(j_k+1)^2}\right)\right),\quad(1)$$
$$j_k\ge 0,\ k=1,\dots,n.$$

We propose the following algorithm for approximating the solutions of the system of equations (1).

Step 1. Find an approximation $\tilde{h}$ of the solution h in terms of the moments m_j, $j\in N^n$. To this end, since $h\in L_\nu^\infty(A)\subset L_\nu^2(A)$, h has a Fourier expansion with respect to the Hilbert base $(\psi_j)_{j_k\ge 0}$ associated following Gram-Schmidt procedure to the complete system of linearly independent polynomials $(\varphi_j)_{j_k\ge 0}$.

The Fourier coefficients $\langle h,\psi_j\rangle$ are given by:

$$\langle h,\psi_j\rangle=\sum_{\substack{l_k\le j_k,\\ k=0,\dots,n}}\alpha_l\langle h,\varphi_l\rangle=\sum_{\substack{l_k\le j_k,\\ k=0,\dots,n}}\alpha_l m_l,$$

where α_l are given by the Gram-Schmidt procedure, so that we know h in terms of the moments. Recall that there exists a subsequence of the sequence of Fourier partial sums, which converges pointwise to $h.$ This fact is a consequence of the remark that the partial sums of the Fourier series converge in an L^2- space. Then one applies theorem 3.12 [21], p. 65. In the sequel we can write: $h \approx \tilde{h},$ where $\tilde{h}$ is a partial sum of the Fourier series of h. Note that all these partial sums are polynomials, so that they are continuous.

Step 2. Let $\tilde{h}$ be a partial sum of the Fourier series with respect to the orthogonal polynomials $(\psi_j)_{j_k \geq 1}$. Using Schwarz inequality, and approximation of continuous functions $\tilde{h}$ by simple functions:

$$\tilde{h}(t_1,\ldots,t_n) \approx \sum_{p,q \leq M} c_{p,q} \chi_{D_{p,q}}(t_1,\ldots,t_n),$$

The numbers $c_{p,q}$ are the values of $\tilde{h}$ at some points in

$$\left\{(t_1,\ldots,t_n); \frac{m_q}{2^p} \leq \tilde{h}(t_1,\ldots,t_n) < \frac{m_q+1}{2^p}\right\},$$

where p is large and m_q is suitable chosen for approximating $\tilde{h}$.

One deduces

$$m_j \approx \int_A \varphi_j \tilde{h}\, d\nu \approx \int_A \varphi_j \left(\sum_{p,q} c_{p,q} \cdot \chi_{D_{p,q}}(t_1,\ldots,t_n)\right) d\nu =$$

$$\sum_{p,q} c_{p,q} \cdot \left(\sum_{m \in N} \int_A \varphi_j \cdot \chi_{[x_{1,m,p,q}, y_{1,m,p,q})}(t_1) \cdots \chi_{[x_{n,m,p,q}, y_{n,m,p,q})}(t_n)\, d\nu\right),$$

where $D_{p,q}$ are open subsets approximating in measure the subsets $\left\{(t_1,\ldots,t_n); \frac{m_q}{2^p} \leq \tilde{h}(t_1,\ldots,t_n) < \frac{m_q+1}{2^p}\right\}$, and whose cell decompositions may be written as below

$$(0,1)^n \supset D_{p,q} = \cup_{m \in \mathrm{N}} [x_{1,m,p,q}, y_{1,m,p,q}) \times \cdots \times [x_{n,m,p,q}, y_{n,m,p,q}) \supset$$
$$\left\{(t_1,\ldots,t_n); \frac{m_q}{2^p} \leq \tilde{h}(t_1,\ldots,t_n) < \frac{m_q+1}{2^p}\right\}.$$

The above arguments yield

$$m_j \approx \sum_{p,q} c_{p,q} \left(\sum_{m\in N} \int_{x_{1,m,p,q}}^{y_{1,m,p,q}} t_1^{j_1}(-\ln t_1)dt_1 \cdots \int_{x_{n,m,p,q}}^{y_{n,m,p,q}} t_n^{j_n}(-\ln t_n)dt_n \right) =$$

$$\sum_{p,q} c_{p,q} \left(\sum_{m\in \mathrm{N}} \prod_{k=1}^{n} \left(-\frac{t_k^{j_k+1} \ln t_k}{j_k+1} \Big|_{x_{k,m,p,q}}^{y_{k,m,p,q}} + \frac{t_k^{j_k+1}}{(j_k+1)^2} \Big|_{x_{k,m,p,q}}^{y_{k,m,p,q}} \right)\right) =$$

$$\sum_{p,q} c_{p,q} \left(\sum_{m\in N} \prod_{k=1}^{n} \left(\frac{x_{k,m,p,q}^{j_k+1} \ln(x_{k,m,p,q}) - y_{k,m,p,q}^{j_k+1} \ln(y_{k,m,p,q})}{j_k+1} + \frac{y_{k,m,p,q}^{j_k+1} - x_{k,m,p,q}^{j_k+1}}{(j_k+1)^2} \right)\right).$$

For the one-dimensional case see [13], Remark 29. The conclusion is that we can determinate (approximate) the "unknowns" $y_{k,m,p,q}, x_{k,m,p,q}, k = 1,\dots,n$ by means of the cell decomposition of the open subsets $D_{p,q}$ associated to the known polynomial $\tilde{h}$ (cf. [21, section 2.19]). The basic relations can be summarized as the system of equations (1), where m_j are given, $c_{p,q}$ are known from Step 1, and the unknowns $x_{k,m,p,q}, y_{k,m,p,q}$ are determined in terms of the cell - decomposition of the suitable chosen open subsets D_{pq}, deduced from the known polynomial $\tilde{h}$ (the measure ν is outer regular [21]). The unknowns are the coordinates of the vertices of the cells (see. [21, section 2.19]). Clearly, the solution is not unique. So, using the above notations, we have proved the following theorem.

Theorem 3.1. *An approximation for the solution of* (1) *is given by the coordinates* $x_{k,m,pq}, y_{k,m,pq}, k \in \{1,\dots,n\}$ *of the vertices of the cells from the cell – decomposition of the open subsets* $D_{p,q}$ *associated to the known polynomials* $\tilde{h}$*.*
For a similar one dimensional problem, having a finite number of unknowns and solved by using other methods see [14].

4. Conclusions

New characterizations for the existence of solutions of abstract and concrete Mazur – Orlicz problems are proved. In the second part of this work, a geometric method of approximating the solution of a system with infinitely many equations and unknowns is sketched. This is a general method. Similar problems can be solved using the same ideas and appropriate modifications. One uses a different method for related problems to those solved in the literature by some other methods.

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