NECESSARY AND SUFFICIENT CONDITIONS FOR IDENTIFYING STRICTLY GEOMETRICALLY $\alpha$-BIDIAGONALLY DOMINANT MATRICES

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In the paper, the authors establish necessary and sufficient conditions for identifying strictly geometrically $\alpha$-bidiagonally dominant matrices, present some new criteria of nonsingular $H$-matrix by using the theory of geometrically $\alpha$-bidiagonally dominant matrices, and provide two numerical examples which illustrate the effectiveness and advantages of the new criteria.

Keywords: strictly geometrically $\alpha$-bidiagonally dominant matrix, nonsingular $H$-matrix, chain of nonzero elements, irreducibility, diagonally dominant matrix


1. Introduction and definitions

Let $\mathbb{C}^{n \times n}$ denote the set of all $n \times n$ complex matrices, $\mathbb{N}$ be the set of all positive integers, and $A = (a_{ij}) \in \mathbb{C}^{n \times n}$. Further let

$$P_i(A) = \sum_{j \neq i} |a_{ij}|, \quad R_i(A) = \sum_{j \neq i} |a_{ji}|, \quad i, j \in \mathbb{N}$$

and

$$M = \{(i, j) : i, j \in \mathbb{N}, i \neq j\} = \mathbb{N}^2 \setminus \{(i, i) : i \in \mathbb{N}\}. \quad \text{(1.2)}$$

Definition 1.1 ([7, Definition 2.1] and [14, p. 427]). Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$. If

$$|a_{ii}| \geq P_i(A) \quad \text{(1.3)}$$

for all $i \in \mathbb{N}$, then $A$ is said to be diagonally dominant and denoted by $A \in D_n$. If the inequality (1.3) is strict for all $i \in \mathbb{N}$, then $A$ is said to be strictly diagonally dominant and denoted by $A \in SD_n$.

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If there exists a positive diagonal matrix

\[ X = \text{diag}(x_1, x_2, \ldots, x_n) \]  

such that \( AX \) is a strictly diagonally dominant matrix, then \( A \) is said to be a generalized strictly diagonally dominant matrix and denoted by \( A \in GSDD_n \).

It is said in\textsuperscript{3,12} and\textsuperscript{4} p. 427 that \( A \) is a nonsingular \( H \)-matrix if and only if \( A \) is a generalized strictly diagonally dominant matrix. It was referenced in\textsuperscript{3} Lemma 3.3 and\textsuperscript{4} p. 241 that, if \( A \) is a nonsingular \( H \)-matrix, then there exists at least one strictly diagonally dominant row.

**Definition 1.2 (\cite{3} Definition 2.2 and \cite{4} Definition 1).** Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \). If there exists \( \alpha \in [0, 1] \) such that \( |a_{ii}| \geq \alpha P_i(A) + (1 - \alpha)R_i(A) \) holds for all \( i \in \mathbb{N} \), then \( A \) is said to be an \( \alpha \)-diagonally dominant matrix. If the inequality is strict, then \( A \) is said to be a strictly \( \alpha \)-diagonally dominant matrix.

If there exists a positive diagonal matrix \( X \) such that \( AX \) is a strictly \( \alpha \)-diagonally dominant matrix, then \( A \) is said to be a generalized strictly \( \alpha \)-diagonally dominant matrix.

**Definition 1.3 (\cite{3} Definition 2).** Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \). If there exists \( \alpha \in [0, 1] \) such that \( |a_{ij}| \geq \alpha P_i(A)P_j(A) + (1 - \alpha)R_i(A)R_j(A) \) holds for all \( (i, j) \in M \), then \( A \) is said to be an \( \alpha \)-bidominated matrix. If the inequality is strict, then \( A \) is said to be a strictly \( \alpha \)-bidominated matrix.

**Definition 1.4 (\cite{3,5}).** Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \). If there exists \( \alpha \in [0, 1] \) such that

\[ |a_{ii}| \geq [P_i(A)]^\alpha [R_i(A)]^{1-\alpha} \]  

for all \( (i, j) \in M \), then \( A \) is called a geometrically \( \alpha \)-bidominated matrix and denoted by \( A \in PDG_n^\alpha \). If the inequality\textsuperscript{3,5} is strict, then \( A \) is said to be a strictly geometrically \( \alpha \)-bidominated matrix and denoted by \( A \in SPDG_n^\alpha \).

**Definition 1.5.** Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \) and

\[ J(A) = \{(i, j) \in M : |a_{ij}a_{jj}| > [P_i(A)P_j(A)]^\alpha [R_i(A)R_j(A)]^{1-\alpha}\}. \]  

1. If there exists an \( \alpha \in [0, 1] \) such that

\[ |a_{ij}a_{jj}| \geq [P_i(A)P_j(A)]^\alpha [R_i(A)R_j(A)]^{1-\alpha} \]  

for all \( (i, j) \in M \), then \( A \) is said to be a geometrically \( \alpha \)-bidominated matrix and denoted by \( A \in DPDG_n^\alpha \).

2. If \( J(A) = M \), then \( A \) is said to be a strictly geometrically \( \alpha \)-bidominated matrix and denoted by \( A \in SPDG_n^\alpha \).

3. If there exists a positive diagonal matrix \( X \) such that \( AX \) is strictly geometrically \( \alpha \)-bidominated, then \( A \) is said to be a generalized strictly geometrically \( \alpha \)-bidominated matrix and denoted by \( A \in GSDPDG_n^\alpha \).
Remark 1.1. If $\alpha = 1$, Definition 1.1 becomes Definition 1.1] in which $A$ is called a (strictly) doubly diagonally dominant matrix.

In [12, p. 1011] and [20, Definition 1], the matrix $A$ defined in Definition 2.3 is respectively called a product $\alpha$-diagonally dominant matrix and an $\alpha$-diagonally dominant matrix. In [1, Definition 2.3], [13, p. 19], [11, p. 88], and [12, p. 1012], the matrix $A$ defined by (1.2) was respectively called an $\alpha$ bi-diagonally dominant matrix, an $\alpha$-connective diagonal dominant matrix, an $\alpha$-bidagonally dominant matrix, and a doubly product $\alpha$-diagonally dominant matrix. It is clear that these four different notations of the same concept confuse us and make us be at loose ends.

It is common knowledge [3, 5] that the geometric mean $G(a,b; \alpha)$ of two positive numbers $a$ and $b$ with unit weight $(\alpha, 1-\alpha)$ for $\alpha \in [0, 1]$ is defined by $a^\alpha b^{1-\alpha}$.

With the help of this notion, the term in the right hand side of (1.1) can be rewritten as $G(P_i(A), R_i(A); \alpha)G(P_j(A), R_j(A); \alpha)$ or $G(P_i(A), R_i(A); \alpha)G(P_j(A), R_i(A); \alpha)$. By this and considering Definitions 2.3, Definition 4.2, and the above mentioned confusion, for consistency and avoiding confusion, we would like to call the matrix defined by (1.1) a geometrically $\alpha$-diagonally dominant matrix and to call the matrix $A$ defined by (1.1) is a geometrically $\alpha$-bidagonally dominant matrix. In our opinion, this terminology is more meaningful and simple. This idea comes from the theory of means [3, 5, 19] and was ever used in generalizations of convex functions, see, for example, [13, 21, 12] and closely related references therein.

In what follows we will use the following notations:

- $M_1(A) = \{(i,j) \in M : P_i(A)P_j(A) < |a_{ij}|, R_i(A)R_j(A)\}$,
- $M_2(A) = \{(i,j) \in M : R_i(A)R_j(A) < |a_{ij}|, P_i(A)P_j(A)\}$,
- $M_3(A) = \{(i,j) \in M : |a_{ij}|, R_i(A)R_j(A) > P_i(A)P_j(A)\}$,
- $M_4(A) = \{(i,j) \in M : |a_{ij}|, P_i(A)P_j(A) > R_i(A)R_j(A)\}$,
- $M_5(A) = \{(i,j) \in M : |a_{ij}|, P_i(A)P_j(A) = R_i(A)R_j(A)\}$,
- $M_6(A) = \{(i,j) \in M : |a_{ij}|, P_i(A)P_j(A) = R_i(A)R_j(A), |a_{ij}|, P_i(A)P_j(A)\}$,

which can be found in [13, p. 427]. It is obvious that

$$M = M_1(A) \cup M_2(A) \cup M_3(A) \cup M_4(A) \cup M_5(A) \cup M_6(A).$$

So far as we know, $H$-matrices have been playing an important role in computational mathematics, control theory, electric system theory, mathematics of economics, and many other fields. See [11, 12]. However, the practical discrimination of an $H$-matrix is very difficult. So it is meaningful to judge whether a matrix is an $H$-matrix or not. In recent years, many scholars have been working on its properties and criterion and have obtained lots of criteria for identifying nonsingular $H$-matrix by using iterative arithmetic and techniques in matrix theory and inequalities, and so on. In [13], some criteria for determining nonsingular $H$-matrices are obtained. For more information, please refer to [1, 2, 3, 4, 5, 12, 12].

In this article, basing on results in [13], according to the theory of geometrically $\alpha$-bidagonally dominant matrix, we will give several necessary and sufficient
conditions for specifying strictly geometrically \(\alpha\)-bidagonally dominant matrices and obtain several new practical criteria for nonsingular \(H\)-matrices. So the criteria for nonsingular \(H\)-matrices is expanded. These results improve and extend some existing ones. Finally, we will show the effectiveness and advantages of the proposed new criteria by two numerical examples.

In what follows, we always assume that \(a_{ii}P_i(A)R_i(A) \neq 0\) for any \(i \in \mathbb{N}\) and that both \(M_1(A)\) and \(M_2(A)\) are not empty, as done in [34, Corollary 1].

2. Lemmas

To attain our aim, we need the following lemmas.

**Lemma 2.1** ([33, Theorem 1]). Let \(A = (a_{ij}) \in \mathbb{C}^{n \times n}\). Then \(A\) is a nonsingular \(H\)-matrix if and only if \(A \in GSDPD^\alpha_n\).

**Lemma 2.2** ([33, Theorem 3] and [34, Lemma 2]). Let \(\alpha \in [0, 1]\), \(A = (a_{ij}) \in DPD^\alpha_n\), and \(G(A) = \{i \in \mathbb{N} : |a_{ii}a_{jj}| > |P_i(A)P_j(A)|^\alpha[R_i(A)R_j(A)]^{1-\alpha}, j \neq i, j \in \mathbb{N}\}.\) Then \(A\) is a nonsingular \(H\)-matrix if \(G(A) \neq \emptyset\) and one of the following two statements is true:

1. when \(\alpha \neq 0\), if \(|a_{ii}a_{jj}| = |P_i(A)P_j(A)|^\alpha[R_i(A)R_j(A)]^{1-\alpha}\) for every \(i, j\) with \(i \neq j\), there exists a nonzero elements chain \(a_{i_0i_1}, a_{i_1i_2}, \ldots, a_{i_{r}j_0}\), where \(i_0 \neq i_1, i_1 \neq i_2, \ldots, i_r \neq j_0\), such that \(i_0 = i\) or \(i_0 = j\) and \(j_0 \in G(A)\);

2. when \(\alpha = 0\), if \(|a_{ii}a_{jj}| = R_i(A)R_j(A)\) for every \(i, j\) with \(i \neq j\), there exists a nonzero elements chain \(a_{j_0j_1}, a_{j_1j_2}, \ldots, a_{j_{r}i_0}\), where \(j_0 \neq j_1, j_1 \neq j_2, \ldots, j_r \neq i_0\), such that \(i_0 = i\) or \(i_0 = j\) and \(j_0 \in G(A)\).

**Lemma 2.3** ([34, Theorem 2]). Let \(A = (a_{ij}) \in \mathbb{C}^{n \times n}\). Then \(A\) is a nonsingular \(H\)-matrix if it satisfies either of the following two conditions:

1. \(A \in SPD^\alpha_n\),

2. \(A \in PD^\alpha_n\) is irreducible and the strict inequality (3.1) holds for at least one.

**Lemma 2.4** ([33, Theorem 2.1 (iv)]). Let \(A = (a_{ij}) \in \mathbb{C}^{n \times n}\) be an irreducible matrix. If \(A \in DPD_1^1\) and if at least one of the inequalities in (3.1) strictly holds, then \(A\) is a nonsingular \(H\)-matrix.

3. Main results

Now we start out to state and prove our main results.

**Theorem 3.1.** Let \(A = (a_{ij}) \in \mathbb{C}^{n \times n}\). Then \(A \in SPD^\alpha_n\) if and only if \(M_0(A) = \emptyset\) and

\[
\frac{\ln|\mathcal{R}_s(A)\mathcal{R}_t(A)| - \ln|a_{ss}a_{tt}|}{\ln|\mathcal{R}_s(A)\mathcal{R}_t(A)| - \ln|\mathcal{P}_s(A)\mathcal{P}_t(A)|} < \frac{\ln|a_{ii}a_{jj}| - \ln|\mathcal{R}_i(A)\mathcal{R}_j(A)|}{\ln|\mathcal{P}_i(A)\mathcal{P}_j(A)| - \ln|\mathcal{R}_i(A)\mathcal{R}_j(A)|}
\]

(3.1)

for all \((s, t) \in M_1(A)\) and \((i, j) \in M_2(A)\).
Proof. We first prove the necessity. Since $A \in SDPD_n^\alpha$ and $M_0(A) = \emptyset$, there exists $\alpha \in [0,1]$ such that $|a_{ss}||a_{tt}| > [P_t(A)P_j(A)]^\alpha[R_s(A)R_t(A)]^{1-\alpha}$ for all $(s,t) \in M_1(A)$. So, we have $\ln[|a_{ss}||a_{tt}|] - \ln[R_s(A)R_t(A)] > \alpha\{\ln[P_t(A)P_j(A)] - \ln[R_s(A)R_t(A)]\}$, that is,

$$\frac{\ln[R_s(A)R_t(A)] - \ln[|a_{ss}||a_{tt}|]}{\ln[R_s(A)R_t(A)] - \ln[P_t(A)P_j(A)]} < \alpha. \tag{3.2}$$

On the other hand, we have $|a_{ij}a_{jj}| > [P_i(A)P_j(A)]^\alpha[R_t(A)R_j(A)]^{1-\alpha}$ for every $(i,j) \in M_2(A)$, which can be reformulated as

$$\ln[P_i(A)P_j(A)] - \ln[|a_{ij}a_{jj}|] < (1-\alpha)\{\ln[P_i(A)P_j(A)] - \ln[R_i(A)R_j(A)]\},$$

that is,

$$\alpha < \frac{\ln[|a_{ij}a_{jj}|] - \ln[R_i(A)R_j(A)]}{\ln[P_i(A)P_j(A)] - \ln[R_i(A)R_j(A)]}. \tag{3.3}$$

Therefore, from (3.2) and (3.3), we obtain (3.1).

Now we begin to prove the sufficiency. Obviously, for all $(s,t) \in M_1(A)$ and $(i,j) \in M_2(A)$, from (3.1), it follows that

$$0 < \frac{\ln[R_s(A)R_t(A)] - \ln[|a_{ss}||a_{tt}|]}{\ln[R_s(A)R_t(A)] - \ln[P_s(A)P_t(A)]} < \frac{\ln[|a_{ij}a_{jj}|] - \ln[R_i(A)R_j(A)]}{\ln[P_i(A)P_j(A)] - \ln[R_i(A)R_j(A)]} < 1.$$ 

Therefore, there must exist $\alpha \in (0,1)$ such that

$$\frac{\ln[R_s(A)R_t(A)] - \ln[|a_{ss}||a_{tt}|]}{\ln[R_s(A)R_t(A)] - \ln[P_s(A)P_t(A)]} < \alpha$$

$$< \frac{\ln[|a_{ij}a_{jj}|] - \ln[R_i(A)R_j(A)]}{\ln[P_i(A)P_j(A)] - \ln[R_i(A)R_j(A)]}. \tag{3.4}$$

The left hand side of (3.1) implies that

$$\ln[|a_{ss}||a_{tt}|] - \ln[R_s(A)R_t(A)] > \alpha\{\ln[P_s(A)P_t(A)] - \ln[R_s(A)R_t(A)]\},$$

i.e., $|a_{ss}||a_{tt}| > [P_s(A)P_t(A)]^\alpha[R_s(A)R_t(A)]^{1-\alpha}$ for $(s,t) \in M_1(A)$. The right hand side of (3.1) means that

$$\frac{\ln[P_i(A)P_j(A)] - \ln[|a_{ij}a_{jj}|]}{\ln[P_i(A)P_j(A)] - \ln[R_i(A)R_j(A)]} < 1 - \alpha,$$

i.e., $|a_{ij}a_{jj}| > [P_i(A)P_j(A)]^\alpha[R_t(A)R_j(A)]^{1-\alpha}$ for $(i,j) \in M_2(A)$. For all $(i,j) \in M_3(A) \cup M_4(A) \cup M_5(A)$ and $\alpha \in (0,1)$, it is easy to see that $|a_{ij}a_{jj}| > [P_i(A)P_j(A)]^\alpha[R_t(A)R_j(A)]^{1-\alpha}$. Since $M_0(A) = \emptyset$, there exists $\alpha \in [0,1]$ such that $|a_{ij}a_{jj}| > [P_i(A)P_j(A)]^\alpha[R_t(A)R_j(A)]^{1-\alpha}$ for $(i,j) \in M$. Consequently, by Definition 2.3, we obtain $A \in SDPD_n^\alpha$. \qed

**Theorem 3.2.** Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ and $M_0(A) = \emptyset$. If $(s,t) \in M_1(A)$ and $(i,j) \in M_2(A)$ satisfy

$$\frac{\ln[R_s(A)R_t(A)] - \ln[|a_{ss}||a_{tt}|]}{\ln[R_s(A)R_t(A)] - \ln[P_s(A)P_t(A)]} < \frac{\ln[|a_{ij}a_{jj}|] - \ln[R_i(A)R_j(A)]}{\ln[P_i(A)P_j(A)] - \ln[R_i(A)R_j(A)]}, \tag{3.5}$$

then $A$ is a nonsingular $H$-matrix.
Proof. From Theorem 2.4, it is clear that \( A \in SDPD^n_\alpha \). Hence, by Lemma 2.4, we obtain that \( A \) is a nonsingular \( H \)-matrix.

Remark 3.1. Since
\[
[P(A)P(A)\alpha[R(A)R(A)]^{1-\alpha} \leq \alpha P(A)P(A) + (1 - \alpha) R(A)R(A)
\]
for \((i, j) \in M(A)\), then \( DD_\alpha \subset SDPD^n_\alpha \), where \( DD_\alpha \) is defined in (3).

Corollary 3.2.1. Let \( A = (a_{ij}) \in C^{n \times n} \). If at least one of the inequalities
\[
\frac{\ln[R(A)r_i(A)] - \ln|a_{ii}|}{\ln[R(A)r_i(A)] - \ln[P(A)r_i(A)]} \leq \frac{\ln[a_{ij}a_{jj}] - \ln[R(A)r_i(A)]}{\ln[P(A)r_i(A)] - \ln[R(A)r_i(A)]}
\]
for \((s, t) \in M_1(A)\) and \((i, j) \in M_2(A)\) hold and \( |a_{rr}||a_{qq}| = P(A)P(A) = R(A)R(A)\) for \((r, q) \in M_6(A) \neq \emptyset\), and if there are nonzero elements \( a_{i1}, a_{i12}, \ldots, a_{ik} \) such that \( l = r \) or \( l = q \), \( k = k_1 \) or \( k = k_2 \), and \((k_1, k_2) \in M - M_6(A)\), then \( A \) is a nonsingular \( H \)-matrix.

Proof. By Theorem 3.2.3, we see that \( |a_{ii}a_{jj}| \geq |P(A)P(A)\alpha[R(A)R(A)]^{1-\alpha}\) for all \((i, j) \in M_1(A) \cup M_2(A) \cup M_3(A) \cup M_4(A) \cup M_5(A)\). On the other hand, for all \((r, q) \in M_6(A) \neq \emptyset\), we obtain \( |a_{rr}||a_{qq}| = P(A)P(A) = R(A)R(A)\) and by the assumption, there exists a nonzero elements chain \( a_{i1}, a_{i12}, \ldots, a_{ik} \) such that \( l = r \) or \( l = q \), \( k = k_1 \) or \( k = k_2 \), and \((k_1, k_2) \in M - M_6(A)\), on the base of the Lemma 3.2.4, we can acquire that \( A \) is a nonsingular \( H \)-matrix.

Corollary 3.2.2. Let \( A = (a_{ij}) \in C^{n \times n} \) be an irreducible matrix. If \( |a_{rr}||a_{qq}| = P(A)P(A) = R(A)R(A)\) for all \((r, q) \in M_6(A)\) and
\[
\frac{\ln[R(A)r_i(A)] - \ln|a_{ii}|}{\ln[R(A)r_i(A)] - \ln[P(A)r_i(A)]} \leq \frac{\ln[a_{ij}a_{jj}] - \ln[R(A)r_i(A)]}{\ln[P(A)r_i(A)] - \ln[R(A)r_i(A)]}
\]
(3.6)
for \((s, t) \in M_1(A)\) and \((i, j) \in M_2(A)\), where at least one of the inequalities in (3.6) strictly holds, then \( A \) is a nonsingular \( H \)-matrix.

Proof. Obviously, according to the assumption, by the similar method in the proof of Corollary 3.2.1, we can obtain \( A \in DPD_\alpha^n \).

For \( \alpha = 0 \) or \( \alpha = 1 \), since \( A \) is irreducible, from Lemma 3.2.1, it follows that \( A \) is a nonsingular \( H \)-matrix.

For \( \alpha \in (0, 1)\), from \( A \in DPD_\alpha^n \), it follows that
\[
|a_{ii}a_{jj}| \geq |P(A)P(A)\alpha[R(A)R(A)]^{1-\alpha}\) (3.7)
for all \((i, j) \in M\). Thus, from \( J(A) \neq \emptyset\), we can easily obtain that there exists at most a number \( i_0 \in \mathbb{N} \) such that \( |a_{i_0i_0}| \leq |P(A)\alpha[R(A)A]^{1-\alpha} |. \) Without loss of generality, we assume that \( |a_{11}| \leq |P(A)\alpha[R(A)A]^{1-\alpha} | \) and \( |a_{jj}| \geq |P(A)\alpha[R(A)A]^{1-\alpha} | \) for \( 2 \leq j \leq n \). Putting \( d^\alpha = |a_{11}|^{-1}[P(A)\alpha[R(A)A]^{1-\alpha} \), by (3.2), we have \( d^\alpha[P(A)\alpha[R(A)A]^{1-\alpha} \leq |a_{jj}| \) for \( 2 \leq j \leq n \). Let \( X = \text{diag}(d, 1, \ldots, 1) \) and \( B = AX = (b_{ij}) \). Then
\[
[P(B)]\alpha[R(B)]^{1-\alpha} = [P(A)]\alpha[R(dA)]^{1-\alpha} = d^\alpha d^{1-\alpha} |a_{11}| = b_{11}
\]
and

\[ [P_j(B)]^\alpha [R_j(B)]^{1-\alpha} = [P_j(dA)]^\alpha [R_1(A)]^{1-\alpha} \]

Thus, we obtain \( B \in PD_n^\alpha \). Since \( B \) is irreducible, from Lemma 1.4, it follows that \( B \) is a nonsingular \( H \)-matrix, and so is \( A \). Corollary 3.2.3 is proved.

**Theorem 3.3.** Let \( A = (a_{ij}) \in C^{n \times n} \). Then \( A \in GSDPD_n^\alpha \) if and only if there exists a positive diagonal matrix \( P \) such that \( M_0(AX) = \emptyset \) and

\[
\frac{\ln[R_s(AX)R_t(AX)] - \ln[a_{ss}x_s||a_{tt}x_t]}{\ln[R_s(AX)R_t(AX)] - \ln[P_s(AX)P_t(AX)]} < \frac{\ln[|a_{ii}x_i||a_{jj}x_j|] - \ln[R_i(AX)R_j(AX)]}{\ln[P_i(AX)P_j(AX)] - \ln[M(AX)R_j(AX)]}
\]

for all \((s, t) \in M_1(AX)\) and \((i, j) \in M_2(AX)\).

**Proof.** By the similar method as in the proof of Theorem 3.1, we can easily prove this theorem. \(\square\)

**Theorem 3.4.** Let \( A = (a_{ij}) \in C^{n \times n} \). If there exists a positive diagonal matrix \( X = \text{diag}(x_1, x_2, \ldots, x_n) \) such that \( M_0(AX) = \emptyset \) and

\[
\frac{\ln[R_s(AX)R_t(AX)] - \ln[a_{ss}x_s||a_{tt}x_t]}{\ln[R_s(AX)R_t(AX)] - \ln[P_s(AX)P_t(AX)]} < \frac{\ln[|a_{ii}x_i||a_{jj}x_j|] - \ln[R_i(AX)R_j(AX)]}{\ln[P_i(AX)P_j(AX)] - \ln[R_i(AX)R_j(AX)]}
\]

for all \((s, t) \in M_1(AX)\) and \((i, j) \in M_2(AX)\), then \( A \) is a nonsingular \( H \)-matrix.

**Proof.** From Theorem 3.3, it is easy to obtain that \( A \in GSDPD_n^\alpha \). As a result, by Lemma 3.1, it follows that \( A \) is a nonsingular \( H \)-matrix. \(\square\)

**Remark 3.2.** According to Theorem 3.3, we can obtain similar results to Corollaries 3.2.1 and 3.2.2.

4. **Two numerical examples**

Finally we provide two numerical examples which illustrate the effectiveness and advantages of the new criteria.

**Example 1.** Let

\[
A = \begin{bmatrix}
2.3 & 1 & 2 \\
1 & 3 & 3 \\
0.5 & 1.5 & 3.5
\end{bmatrix}
\]

An easy computation yields

\[
\frac{R_2(A)R_3(A) - |a_{22}||a_{33}|}{R_2(A)R_3(A) - P_2(A)R_3(A)} = \frac{4}{9} > \frac{|a_{11}||a_{22}| - R_1(A)R_2(A)}{P_1(A)P_2(A) - R_1(A)R_2(A)}
\]

Thus \( A \) does not satisfy conditions of [13, Theorem 2].
By the criteria presented in Theorem 42, we obtain
\[
0.3906 \cdots = \frac{\ln|R_2(A)R_3(A)| - \ln|a_{22}|a_{33}|}{\ln|P_2(P_2(A)) - \ln|P_3(P_3(A))|} < \frac{\ln|a_{11}|a_{22}| - \ln|R_1(A)R_2(A)|}{\ln|P_1(P_1(A)) - \ln|R_1(A)R_2(A)|} = 0.5242 \cdots.
\]
Since \(M_6(A) = \emptyset\), the matrix \(A\) satisfies conditions of Theorem 42, then \(A\) is a nonsingular \(H\)-matrix. Let \(X = \text{diag}(1,1.04,0.6)\), then \(AX \in SD_n\). \(\square\)

**Example 2.** Let
\[
A = \begin{bmatrix}
0.3 & 1 & 2 \\
0.125 & 3 & 3 \\
0.0625 & 1.5 & 3.5
\end{bmatrix}.
\]

Since
\[
\frac{R_2(A)R_3(A) - |a_{22}|a_{33}|}{R_2(A)R_3(A) - P_2(A)P_3(A)} = 0.2625 \cdots > \frac{|a_{11}|a_{22}| - R_1(A)R_2(A)}{P_1(A)P_2(A) - R_2(A)R_3(A)} = 0.0484 \cdots > \frac{|a_{11}|a_{33}| - R_1(A)R_3(A)}{P_1(A)P_3(A) - R_1(A)R_3(A)} = 0.0300 \cdots,
\]
and
\[
\frac{\ln|a_{11}|a_{22}| - \ln|R_1(A)R_2(A)|}{\ln|P_1(A)P_2(A)| - \ln|R_1(A)R_2(A)|} = 0.2177 \cdots
\]
\[
> \frac{\ln|R_2(A)R_3(A)| - \ln|a_{22}|a_{33}|}{\ln|R_2(A)R_3(A)| - \ln|P_2(A)P_3(A)|} = 0.1854 \cdots
\]
\[
> \frac{\ln|a_{11}|a_{33}| - \ln|R_1(A)R_3(A)|}{\ln|P_1(A)P_3(A)| - \ln|R_1(A)R_3(A)|} = 0.0704 \cdots,
\]
and since \(M_6(A) = \emptyset\), the matrix \(A\) does not satisfy the conditions in [41, Theorem 2] and Theorem 42 in this paper. However, if we choose the positive diagonal matrix \(X = \text{diag}(8,1,1)\), then
\[
AX = \begin{bmatrix}
2.4 & 1 & 2 \\
1 & 3 & 3 \\
0.5 & 1.5 & 3.5
\end{bmatrix}.
\]

Since
\[
\frac{\ln|R_2(A)AX|R_3(A)| - \ln|a_{22x2}|a_{33x3}|}{\ln|R_2(A)AX|R_3(A)| - \ln|P_2(A)X|P_3(A)|} = 0.3906 \cdots
\]
\[
< \frac{\ln|a_{11x1}|a_{22x2}| - \ln|R_1(A)AX|R_2(A)|}{\ln|P_1(A)X|P_2(A)| - \ln|R_1(A)AX|R_2(A)|} = 0.5608 \cdots,
\]
the matrix \(A\) satisfies conditions of Theorem 43 in this paper, and then \(A\) is a nonsingular \(H\)-matrix. In fact, if we take the positive diagonal matrix \(Y = \text{diag}(1,0.13,0.08)\), then \(AY \in SD_n\). \(\square\)
5. Conclusions

In conclusion, we unify the notion “the geometrically $\alpha$-bidiagonally dominant matrix”, establish necessary and sufficient conditions for identifying strictly geometrically $\alpha$-bidiagonally dominant matrices, and present some new criteria of nonsingular $H$-matrix.

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