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APPROXIMATE OPTIMAL SOLUTIONS AND GENERALIZED CONTRACTIONS IN THE SENSE OF CHATTERJEA

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In the current paper, we introduce a new class of non-self mappings, called proximal generalized contractions. We provide different existence, uniqueness and convergence results of an optimal solution for a nonlinear programming problem. In this way, we obtain a new best proximity point theorem and hence we conclude a real extension of Chatterjea's fixed point theorem as a result.

Keywords: nonlinear programming problem; best proximity point; fixed point.
MSC2010: 47H10, 47H09.

1. Introduction

The Banach contraction principle plays a very important role in nonlinear analysis and has many generalizations; see [18, 32] and references therein. Recently, Suzuki [34] established the following fixed point theorem, which is a new type of extension of the Banach contraction principle and does characterize the metric completeness.

Theorem 1.1. ([34]) *Define a nondecreasing function* $\theta : [0,1) \to (\frac{1}{2}, 1]$ *by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le \frac{1}{2}(\sqrt{5}-1), \\ \frac{1-r}{r^2} & \text{if } \frac{1}{2}(\sqrt{5}-1) \le r \le \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r} & \text{if } \frac{1}{\sqrt{2}} \le r < 1. \end{cases} \tag{1.1}$$

Then for a metric space (X, d)*, the following are equivalent:*
(i) X *is complete.*
(ii) Every mapping T *on* X *satisfying the following has a fixed point:*
• *There exists* $r \in [0,1)$ *such that* $\theta(r)d(x,Tx) \le d(x,y)$ *implies* $d(Tx,Ty) \le rd(x,y)$ *for all* $x, y \in X$.
(iii) There exists $r \in (0,1)$ *such that every mapping* T *on* X *satisfying the following has a fixed point:*
• $\frac{1}{10000}d(x,Tx) \le d(x,y)$ *implies* $d(Tx,Ty) \le rd(x,y)$ *for all* $x, y \in X$.

Remark 1.1. Note that for every $r \in [0,1), \theta(r)$ is the best constant.

In 1972, Chatterjea [8] introduced the following notion of contractive type condition for self-mappings.

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Definition 1.1. Let (X, d) be a metric space. A self-mapping $T : X \longrightarrow X$ is called *Chatterjea contraction* if there exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)],$$

for all $x, y \in X$.

We know that if X is complete metric space, every Chatterjea contraction self-mapping defined on X has a unique fixed point ([8]). Note that, the Chatterjea contraction self-mappings may not be continuous. In this regarding, several fixed point results have been given in the literature (see for instance [7, 19]).

2. **Preliminaries**

Consider the non-self mapping $T : A \to X$, in which A is a nonempty subset of a metric space (X, d). Clearly, the fixed point equation $Tx = x$ may not have solution. Hence, it is contemplated to find an element $x \in A$ such that the error $d(x, Tx)$ is minimum. Indeed, best approximation theory has been derived from this idea. Here, we state the following well-known best approximation theorem due to Ky Fan.

Theorem 2.1. ([13]) *Let A be a nonempty compact convex subset of a normed linear space X and $T : A \to X$ be a continuous function. Then there exists $x \in A$ such that*

$$\|x - Tx\| = dist(Tx, A) := \inf\{\|Tx - a\| : a \in A\}.$$

Let A and B be nonempty subsets of a metric space (X, d) and $T : A \to B$ be a non-self mapping. A point $x^\star \in A$ is called a *best proximity point* of T if

$$d(x^\star, Tx^\star) = dist(A, B) := \{d(x, y) : (x, y) \in A \times B\}.$$

In fact, best proximity point theorems have been studied to find necessary conditions such that the minimization problem

$$\min_{x \in A} d(x, Tx), \tag{2.1}$$

has at least one solution.

Best proximity point theory is an interesting subject of optimization theory which recently attracted the attention of many authors (see for instance [2, 3, 9, 10, 11, 12, 15, 17, 16, 20, 21, 22, 24, 25, 26, 27, 31, 28, 33, 35]). For other related results, we refer to [4, 5].

Let A, B be two nonempty subsets of a metric space (X, d). Let us fix the following notation which will be needed throughout this article:

$$A_0 := \{x \in A : d(x, y) = dist(A, B) \quad \text{for some} \quad y \in B\},$$

$$B_0 := \{y \in B : d(x, y) = dist(A, B) \quad \text{for some} \quad x \in A\},$$

$$d^*(a, b) := d(a, b) - dist(A, B) \quad \text{for each} \quad (a, b) \in A \times B.$$

It is easy to see that if (A, B) is a nonempty and weakly compact pair of subsets of a Banach space X, then A_0 and B_0 are nonempty subsets of X.

The notion of *proximal contractions* was defined by Sadiq Basha, as follows.

Definition 2.1. ([23]) Let (A, B) be a pair of nonempty subsets of a metric space (X, d). A mapping $T : A \longrightarrow B$ is said to be a proximal contraction if there exists a non-negative real number $\alpha < 1$ such that, for all $u_1, u_2, x_1, x_2 \in A$,

$$\begin{cases} d(u_1, Tx_1) = dist(A, B) \\ d(u_2, Tx_2) = dist(A, B) \end{cases} \Rightarrow d(u_1, u_2) \leq \alpha d(x_1, x_2).$$

To state the main result of [23], we recall the following notion.

Definition 2.2. ([23]) Let A, B be two nonempty subsets of a metric space (X, d). A is said to be approximatively compact with respect to B if every sequence $\{x_n\}$ of A satisfying the condition that $d(y, x_n) \longrightarrow D(y, A)$ for some $y \in B$ has a convergent subsequence.

The next theorem guarantees the existence and uniqueness of a best proximity point for proximal contractions.

Theorem 2.2. ([23]) *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty and B is approximatively compact with respect to A. Assume that $T : A \to B$ is a proximal contraction such that $T(A_0) \subseteq B_0$. Then T has a unique best proximity point.*

There have been many subsequent extensions of Theorem 2.4 due to Sadiq Basha, see [6, 14, 29].

In this article, let us consider a self mapping $g : A \longrightarrow A$ and a non-self mapping $T : A \longrightarrow B$, where (A, B) is a nonempty pair of subsets of a metric space (X, d). We consider the following nonlinear programming problem: Find

$$\min_{x \in A} d(gx, Tx), \tag{2.2}$$

where T belongs to a new class of non-self mappings. We say that a point $x^\star \in A$ is a solution of (2.2) provided that $d(gx^\star, Tx^\star) = dist(A, B)$. As a special case, if g is an identity mapping defined on A, then existence of the solution of (2.2) is equivalent to the existence of a best proximity point for the non-self mapping T. As a corollary of our discussion, we present a new fixed point theorem for generalized Chatterjea contractions with the constant in complete metric spaces.

3. **Main result**

To establish our main results of this section, we introduce the following new class of non-self mappings.

Definition 3.1. Define a strictly decreasing function η from $[0, \frac{1}{2})$ onto $(\frac{1}{2}, 1]$ by

$$\eta(r) := 1 - r.$$

Let A, B be two nonempty subsets of a metric space (X, d). Let $\alpha \in [0, \frac{1}{2})$ and put $r := \frac{\alpha}{1-\alpha}$. A non-self mapping $T : A \longrightarrow B$ is said to be a proximal generalized Chatterjea contraction if for all $u, v, x, y \in A$ with

$$d(u, Tx) = dist(A, B) \text{ and } d(v, Ty) = dist(A, B),$$

we have

$$\eta(r) d^*(x, Tx) \leq d(x, y) \text{ implies } d(u, v) \leq \alpha[d^*(x, Ty) + d^*(y, Tx)]. \tag{3.1}$$

The notion of *a proximal Chatterjea contraction* can be defined as follows.

Definition 3.2. Let A, B be two nonempty subsets of a metric space (X, d). A non-self mapping $T : A \longrightarrow B$ is said to be a proximal Chatterjea contraction if there exists $\alpha \in [0, \frac{1}{2})$ such that for all $u, v, x, y \in A$ with

$$d(u, Tx) = dist(A, B) \text{ and } d(v, Ty) = dist(A, B),$$

we have

$$d(u, v) \leq \alpha[d^*(x, Ty) + d^*(y, Tx)].$$

Note that the class of proximal generalized Chatterjea contractions contains the class of proximal Chatterjea contractions as a subclass. Also, it is clear that the class of proximal Chatterjea contractions contains the class of Chatterjea contraction non-self mappings.

Here, we establish the main result of this section which ensures the existence and uniqueness of a solution of the nonlinear programming problem (2.1).

Theorem 3.1. *Let (A, B) be a nonempty pair of subsets of a complete metric space (X, d) such that A_0 is nonempty and closed. Assume that $T : A \to B$ is a proximal generalized Chatterjea contraction such that $T(A_0) \subseteq B_0$. Then T has a unique best proximity point. Moreover, if $\{x_n\}$ is a sequence in A such that $d(x_{n+1}, Tx_n) = dist(A, B)$ then $\{x_n\}$ converges to the best proximity point of T.*

Proof. Assume $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that $d(x_1, Tx_0) = dist(A, B)$. Again, since Tx_1 is a member of $T(A_0)$ which is a subset of B_0, it follows that there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = dist(A, B)$. Continuing this process, we can find a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = dist(A, B), \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

By using the relation (3.2), we conclude that

$$d(x_0, Tx_0) \leq d(x_0, x_1) + d(x_1, Tx_0) = d(x_0, x_1) + dist(A, B).$$

Since for each $r \in [0, \frac{1}{2})$, $\eta(r) \leq 1$ we have

$$\eta(r)d^*(x_0, Tx_0) \leq d^*(x_0, Tx_0) \leq d(x_0, x_1) \ \& \ \begin{cases} d(x_1, Tx_0) = dist(A, B), \\ d(x_2, Tx_1) = dist(A, B). \end{cases}$$

Since T is a proximal generalized Chatterjea contraction non-self mapping, we deduce that

$$\begin{aligned} d(x_1, x_2) &\leq \alpha[d^*(x_0, Tx_1) + d^*(x_1, Tx_0)] \\ = \alpha d^*(x_0, Tx_1) &\leq \alpha[d(x_0, x_1) + d(x_1, x_2) + d^*(x_2, Tx_1)] \\ &= \alpha[d(x_0, x_1) + d(x_1, x_2)]. \end{aligned}$$

Hence,

$$d(x_1, x_2) \leq \frac{\alpha}{1 - \alpha} d(x_0, x_1) = rd(x_0, x_1).$$

It follows from the similar argument that

$$\eta(r)d^*(x_1, Tx_1) \leq d(x_1, x_2) \ \& \ \begin{cases} d(x_2, Tx_1) = dist(A, B), \\ d(x_3, Tx_2) = dist(A, B). \end{cases}$$

Therefore,

$$\begin{aligned} d(x_2, x_3) &\le \alpha[d^*(x_1, Tx_2) + d^*(x_2, Tx_1)] \\ &\le \alpha[d(x_1, x_2) + d(x_2, x_3) + d^*(x_3, Tx_2)] \\ &= \alpha[d(x_1, x_2) + d(x_2, x_3)]. \end{aligned}$$

Thus,

$$d(x_2, x_3) \le \frac{\alpha}{1-\alpha} d(x_1, x_2) = rd(x_1, x_2) \le r^2 d(x_0, x_1).$$

By induction, we conclude that

$$d(x_n, x_{n+1}) \le r^n d(x_0, x_1).$$

Therefore,

$$\Sigma_{n=1}^{\infty} d(x_n, x_{n+1}) \le \Sigma_{n=1}^{\infty} r^n d(x_0, x_1) < \infty.$$

So, $\{x_n\}$ is a Cauchy sequence in A_0. Since A_0 is closed and X is complete metric space, we deduce that $\{x_n\}$ is a convergent sequence. Let $x^\star \in A_0$ be such that $x_n \longrightarrow x^\star$. We assert that $x^\star$ is a unique best proximity point of T. We prove that

$$d^*(x^\star, Tx) \le rd(x^\star, x), \quad \forall x \in A_0 \quad \text{with} \quad x \neq x^\star. \tag{3.3}$$

Suppose $x \in A_0$ and $x \neq x^\star$. Since $T(A_0) \subseteq B_0$, there exists an element y in A_0 such that

$$d(y, Tx) = dist(A, B).$$

Since $x_n \longrightarrow x^\star$, there exists $N_1 \in \mathbb{N}$ such that $d(x_n, x^\star) \le \frac{1}{3} d(x, x^\star)$ for all $n \ge N_1$. For each $n \ge N_1$,

$$\begin{aligned} \eta(r)d^*(x_n, Tx_n) &\le d^*(x_n, Tx_n) \\ &\le d(x_n, x^\star) + d(x^\star, x_{n+1}) + d^*(x_{n+1}, Tx_n) \\ &= d(x_n, x^\star) + d(x^\star, x_{n+1}) \le \frac{2}{3} d(x, x^\star) \\ &= d(x, x^\star) - \frac{1}{3} d(x, x^\star) \le d(x, x^\star) - d(x_n, x^\star) \le d(x_n, x). \end{aligned}$$

This implies that

$$\eta(r)d^*(x_n, Tx_n) \le d(x_n, x) \quad \text{and} \quad \begin{cases} d(x_{n+1}, Tx_n) = dist(A, B), \\ d(y, Tx) = dist(A, B). \end{cases}$$

Since T is a proximal generalized Chatterjea contraction, we conclude that

$$d(x_{n+1}, y) \le \alpha[d^*(x_n, Tx) + d^*(x, Tx_n)] \le \alpha[d^*(x_n, Tx) + d(x, x_{n+1})].$$

Thus,

$$\begin{aligned} d(x^\star, Tx) &= \lim_{n \longrightarrow \infty} d(x_{n+1}, Tx) \\ &\le \lim_{n \longrightarrow \infty} [d(x_{n+1}, y) + d(y, Tx)] \\ &\le \lim_{n \longrightarrow \infty} [\alpha(d^*(x_n, Tx) + d(x, x_{n+1})) + dist(A, B)] \\ &= \alpha d^*(x^\star, Tx) + \alpha d(x^\star, x) + dist(A, B). \end{aligned}$$

Therefore,

$$(1-\alpha)d^*(x^\star, Tx) \le \alpha d(x^\star, x),$$

and so,

$$d^*(x^\star, Tx) \le rd(x^\star, x), \quad \forall x \in A_0, \text{ with } x \neq x^\star,$$

that is, (3.3) holds. We now have

$$\begin{aligned} d^*(x_n, Tx_n) &\leq d(x_n, x^\star) + d^*(x^\star, Tx_n) \\ &\leq d(x_n, x^\star) + rd(x^\star, x_n), \end{aligned}$$

which deduces that

$$\eta(r)d^*(x_n, Tx_n) \leq \frac{1}{1+r} d^*(x_n, Tx_n) \leq d(x^\star, x_n).$$

Besides, since $x^\star \in A_0$ and $T(A_0) \subseteq B_0$, there exists $y^\star \in B_0$ such that $d(y^\star, Tx^\star) = dist(A, B)$. Then

$$\eta(r)d^*(x_n, Tx_n) \leq d(x^\star, x_n) \quad \text{and} \quad \begin{cases} d(x_{n+1}, Tx_n) = dist(A, B), \\ d(y^\star, Tx^\star) = dist(A, B), \end{cases}$$

and hence,

$$\begin{aligned} d(x_{n+1}, y^\star) &\leq \alpha[d^*(x_n, Tx^\star) + d^*(x^\star, Tx_n)] \\ &\leq \alpha[d^*(x_n, Tx^\star) + d^*(x^\star, x_{n+1})]. \end{aligned}$$

Letting $n \longrightarrow \infty$, we obtain

$$\begin{aligned} d(y^\star, x^\star) &\leq \alpha d^*(x^\star, Tx^\star) \\ &\leq \alpha[d(x^\star, y^\star) + d^*(y^\star, Tx^\star)] = \alpha d(x^\star, y^\star). \end{aligned}$$

So, we must have $d(x^\star, y^\star) = 0$ or $x^\star = y^\star$. Hence, $x^\star$ is a best proximity point of the mapping T. The uniqueness of best proximity point follows from the fact that T is a proximal generalized Chatterjea contraction. Indeed, if $x_1^\star, x_2^\star$ are two distinct points in A_0 such that $d(x_i^\star, Tx_i^\star) = dist(A, B)$, for $i = 1, 2$. then

$$\eta(r)d^*(x_1^\star, Tx_1^\star) \leq d(x_1^\star, x_2^\star) \quad \text{and} \quad \begin{cases} d(x_1^\star, Tx_1^\star) = dist(A, B), \\ d(x_2^\star, Tx_2^\star) = dist(A, B), \end{cases}$$

Therefore,

$$\begin{aligned} 0 < d(x_1^\star, x_2^\star) &\leq \alpha[d^*(x_1^\star, Tx_2^\star) + d^*(x_2^\star, Tx_1^\star)] \\ &\leq \alpha[d(x_1^\star, x_2^\star) + d(x_2^\star, x_1^\star)] = 2\alpha d(x_1^\star, x_2^\star) \\ &< d(x_1^\star, x_2^\star), \end{aligned}$$

which is a contradiction. Hence, the best proximity point of T is unique and this completes the proof.

□

We now conclude the next corollary from Theorem 3.3, immediately.

Corollary 3.1. *Let (A, B) be a nonempty pair of a complete metric space (X, d) such that A_0 is nonempty and closed. Assume that $T : A \to B$ is a proximal Chatterjea contraction non-self mapping such that $T(A_0) \subseteq B_0$. Then there exists a unique point $x^\star \in A$ such that $d(x^\star, Tx^\star) = dist(A, B)$. Moreover, if $\{x_n\}$ is a sequence in A_0 such that $d(x_{n+1}, Tx_n) = dist(A, B)$, then $x_n \longrightarrow x^\star$.*

As a result of Theorem 3.3, we obtain the following new fixed point theorem, which is a real extension of Chatterjea's fixed point theorem.

Corollary 3.2. *Let A be a nonempty and closed subset of a complete metric space (X, d). Assume that $T : A \to A$ is a self mapping such that*

$$\eta(r)d(x,Tx) \leq d(x,y) \quad \textit{implies} \quad d(Tx,Ty) \leq \alpha[d(x,Ty) + d(y,Tx)],$$

for all $x, y \in A$. Then T has a unique fixed point $x^\star \in A$. Moreover, if $x_0 \in A$ and we define $x_{n+1} := Tx_n$, then $x_n \longrightarrow x^\star$.

Corollary 3.3. *(Chatterjea fixed point theorem) Let A be a nonempty and closed subset of a complete metric space (X, d). Assume that $T : A \to A$ is a Chatterjea contraction mapping. Then T has a unique fixed point. Moreover, for each $x_0 \in A$, if we define $x_{n+1} := Tx_n$ then the sequence $\{x_n\}$ converges to the fixed point of T.*

Example 3.1. Suppose that $X := \mathbb{R}$ with the usual metric. Let

$$A := [-1, 1] \cup \{4\} \quad \& \quad B := [2, 3].$$

Then A and B are nonempty closed subsets of X and $A_0 = \{1, 4\}$ and $B_0 = \{2, 3\}$. We note that $dist(A, B) = 1$. Let $T : A \longrightarrow B$ be a mapping defined as

$$T(x) = \begin{cases} \frac{5}{2} & \text{if} \quad x = 0, \\ 3 & \text{if} \quad x \neq 0. \end{cases}$$

We can see that T is a proximal generalized Chatterjea contraction non-self mapping for each $\alpha \in [0, \frac{1}{2})$. Indeed, it is sufficient to note that if $d(u, Tx) = dist(A, B)$, then we must have $u = 4$ and $x \in A - \{0\}$. Now, Theorem 3.3 guaranties the existence and uniqueness of a best proximity point for T and this point is $x^* = 4$.

4. **Additional results**

In this section, we establish the existence and uniqueness of solution for the nonlinear programming problem (2.2) under sufficient conditions. We begin our main results of this section with the following geometric property in metric spaces.

Definition 4.1. ([15]) Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. The pair (A, B) is said to have WP-property if and only if

$$\begin{cases} d(x_1, y_1) = dist(A, B) \\ d(x_2, y_2) = dist(A, B) \end{cases} \Rightarrow d(x_1, x_2) \leq d(y_1, y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Let us illustrate this subject with the next example.

Example 4.1. Consider $X := \mathbb{R}$ with the usual metric. Suppose that

$$A : [1, 2], \quad \text{and} \quad B := [-1, 0] \cup \{3\}.$$

It is clear that $dist(A, B) = 1$ and $A_0 = \{1, 2\}$ and $B_0 = \{0, 3\}$. If $(x_1, x_2) = (1, 2)$ and $(y_1, y_2) = (0, 3)$, we have

$$d(x_1, y_1) = d(x_2, y_2) = dist(A, B) \quad \text{and } d(x_1, x_2) < d(y_1, y_2).$$

Thereby, the pair (A, B) has the WP-property. Note that (B, A) has not WP-property.

We also recall that every nonempty, closed convex pair of subsets of a Hilbert space $\mathbb{H}$ has the WP-property ([30]). Moreover, in the setting of uniformly convex

Banach space X, every nonempty, bounded, closed and convex pair of subsets of X has the WP-property ([1]).

Here, we state the main result of this section.

Theorem 4.1. *Let (A, B) be a pair of nonempty, closed subsets of a complete metric space (X, d) such that A_0 is nonempty and (A, B) has the WP-property. Assume that $T : A \to B$ and $g : A \longrightarrow A$ satisfy the following conditions:*

(i) There exists $r \in [0, \frac{1}{2})$ such that

$$\eta(r)d^*(gx, Tx) \leq d(x, y) \quad \textit{implies} \quad d(Tx, Ty) \leq \alpha[d^*(gx, Ty) + d^*(gy, Tx)], \tag{4.1}$$

for each $x, y \in A$.

(ii) $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$.

(iii) g is an isometry.

Then there exists a unique element $x^\star \in A_0$ such that

$$d(gx^\star, Tx^\star) = dist(A, B).$$

Moreover, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$, defined by $d(gx_{n+1}, Tx_n) = dist(A, B)$, converges to the element $x^\star$.

Proof. Let x_0 be a fixed element. Since $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists an element $x_1 \in A_0$ such that $d(gx_1, Tx_0) = dist(A, B)$. Again, since Tx_1 is a member of $T(A_0)$ which is a subset of B_0 and A_0 is a subset of $g(A_0)$ it follows that there exists an element $x_2 \in A_0$ such that $d(gx_2, Tx_1) = dist(A, B)$. By the similar argument, we obtain a sequence $\{x_n\}$ in A_0 such that

$$d(gx_{n+1}, Tx_n) = dist(A, B), \quad \forall n \in \mathbb{N}. \tag{4.2}$$

By the fact that (A, B) has the WP-property and that g is an isometry, we conclude that

$$d(x_{n+1}, x_n) = d(gx_{n+1}, gx_n) \leq d(Tx_n, Tx_{n-1}), \quad \forall n \in \mathbb{N}. \tag{4.3}$$

Now, for all $n \in \mathbb{N} \cup \{0\}$ we have

$$d(gx_n, Tx_n) \leq d(gx_n, gx_{n+1}) + d(gx_{n+1}, Tx_n) = d(x_n, x_{n+1}) + dist(A, B),$$

which implies that

$$\eta(r)d^*(gx_n, Tx_n) \leq d(x_n, x_{n+1}).$$

By using (4.1) we conclude that

$$\begin{aligned} d(x_{n+2}, x_{n+1}) \leq d(Tx_{n+1}, Tx_n) &\leq \alpha[d^*(gx_n, Tx_{n+1}) + d^*(gx_{n+1}, Tx_n)] \\ &\leq \alpha[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + d^*(gx_{n+2}, Tx_{n+1})] \\ &= \alpha[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]. \end{aligned}$$

Now, we obtain

$$d(x_{n+1}, x_{n+2}) \leq rd(x_n, x_{n+1}), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

This implies that $\{x_n\}$ is a Cauchy sequence in A. Since X is complete metric space and A is closed, there exists an element $x^\star \in A$ such that $x_n \longrightarrow x^\star$. We now prove that

$$d^*(gx^\star, Tx) \leq rd(x^\star, x), \quad \forall x \in A \quad \text{with} \quad x \neq x^\star. \tag{4.4}$$

Since $x_n \to x^\star$, there exists $N_1 \in \mathbb{N}$ such that

$$d(x_n, x^\star) \leq \frac{1}{3} d(x, x^\star) \quad \text{for all} \quad n \geq N_1.$$

We have

$$\begin{aligned} \eta(r) d^*(gx_n, Tx_n) \leq d^*(gx_n, Tx_n) &\leq d(gx_n, gx_{n+1}) + d^*(gx_{n+1}, Tx_n) \\ &\leq d(gx_n, gx^\star) + d(gx_{n+1}, gx^\star) \\ &= d(x_n, x^\star) + d(x^\star, x_{n+1}) \\ &\leq \frac{2}{3} d(x, x^\star) = d(x, x^\star) - \frac{1}{3} d(x, x^\star) \\ &\leq d(x, x^\star) - d(x_n, x^\star) \leq d(x_n, x). \end{aligned}$$

Hence,

$$d(Tx_n, Tx) \leq \alpha[d^*(gx_n, Tx) + d^*(gx, Tx_n)], \quad \forall n \geq N_1.$$

Therefore,

$$\begin{aligned} d^*(gx^\star, Tx) &= \lim_{n \longrightarrow \infty} d^*(gx_{n+1}, Tx) \\ &\leq \lim_{n \longrightarrow \infty} [d^*(gx_{n+1}, Tx_n) + d(Tx_n, Tx)] \\ &\leq \lim_{n \longrightarrow \infty} \alpha[d^*(gx_n, Tx) + d^*(gx, Tx_n)] \\ &\leq \lim_{n \longrightarrow \infty} \alpha[d^*(gx_n, Tx) + d(gx, gx_{n+1}) + d^*(gx_{n+1}, Tx_n)] \\ &= \alpha[d^*(gx^\star, Tx) + d(x, x^\star)], \end{aligned}$$

which concludes that

$$d^*(gx^\star, Tx) \leq r d(x^\star, x), \quad \forall x \in A \quad \text{with} \quad x \neq x^\star,$$

that is, (4.4) holds. So,

$$\begin{aligned} d^*(gx_n, Tx_n) &\leq d(gx_n, gx^\star) + d^*(gx^\star, Tx_n) \\ &\leq d(x_n, x^\star) + r d(x^\star, x_n) = (1+r) d(x_n, x^\star). \end{aligned}$$

Thus,

$$\eta(r) d^*(gx_n, Tx_n) \leq \frac{1}{1+r} d^*(gx_n, Tx_n) \leq d(x_n, x^\star),$$

which deduces that

$$\begin{aligned} d(Tx_n, Tx^\star) &\leq \alpha[d^*(gx_n, Tx^\star) + d^*(Tx_n, gx^\star)] \\ &\leq \alpha[d^*(gx_n, Tx_n) + d(Tx_n, Tx^\star) + d(x_{n+1}, x^\star)] \\ &\leq \alpha[(1+r) d(x_n, x^\star) + d(Tx_n, Tx^*) + d(x_{n+1}, x^\star)]. \end{aligned}$$

Hence,

$$d(Tx_n, Tx^\star) \leq r[(1+r) d(x_n, x^\star) + d(x_{n+1}, x^\star)].$$

Letting $n \longrightarrow \infty$ in above relation, we obtain $Tx_n \longrightarrow Tx^\star$. Then

$$d(gx^\star, Tx^\star) = \lim_{n \longrightarrow \infty} d(gx_{n+1}, Tx_n) = dist(A, B),$$

that is, $x^\star$ is a solution of nonlinear programming problem (2.2). If $y^\star \in A_0$ is another solution of (2.2) then we must have

$$d(gy^\star, Ty^\star) = dist(A, B).$$

Since (A,B) has the WP-property, we obtain

$$d(x^\star, y^\star) = d(gx^\star, gy^\star) \leq d(Tx^\star, Ty^\star).$$

We now have

$$\eta(r)d^*(gx^\star, Tx^\star) = 0 \leq d(x^\star, y^\star).$$

Then,

$$d(x^\star, y^\star) \leq d(Tx^\star, Ty^\star) \leq \alpha[d^*(gx^\star, Tx^\star) + d^*(gy^\star, Ty^\star)] = 0,$$

which concludes that $x^\star = y^\star$ and this completes the proof.

□

If in Theorem 4.2 g is an identity mapping, we get the following best proximity point results.

Corollary 4.1. *Let (A,B) be a pair of nonempty, closed subsets of a complete metric space (X,d) such that A_0 is nonempty and (A,B) has the WP-property. Assume that $T : A \to B$ is a generalized Chatterjea contraction non-self mapping, that is, there exists $r \in [0, \frac{1}{2})$ such that*

$$\eta(r)d^*(x, Tx) \leq d(x,y) \quad \textit{implies} \quad d(Tx, Ty) \leq \alpha[d^*(x, Ty) + d^*(y, Tx)], \tag{4.5}$$

for each $x, y \in A$. If $T(A_0) \subseteq B_0$, then there exists a unique point $x^\star \in A_0$ such that

$$d(x^\star, Tx^\star) = dist(A,B).$$

Moreover, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$, defined by $d(x_{n+1}, Tx_n) = dist(A,B)$, converges to the element $x^\star$.

Corollary 4.2. *Let (A,B) be a pair of nonempty, bounded, closed and convex subsets of a uniformly convex Banach space X. Assume that $T : A \to B$ is a generalized Chatterjea contraction non-self mapping such that $T(A_0) \subseteq B_0$. Then there exists a unique point $x^\star \in A_0$ such that*

$$\|x^\star - Tx^\star\| = dist(A,B).$$

Moreover, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$, defined by $\|x_{n+1} - Tx_n\| = dist(A,B)$, converges to the element $x^\star$.

Corollary 4.3. *Let (A,B) be a pair of nonempty, closed subsets of a complete metric space (X,d) such that A_0 is nonempty and (A,B) has the WP-property. Assume that $T : A \to B$ is non-self mapping such that $T(A_0) \subseteq B_0$. If there exists $\alpha \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq \alpha[d^*(x, Ty) + d^*(y, Tx)],$$

for each $x, y \in A$, then there exists a unique point $x^\star \in A_0$ such that

$$d(x^\star, Tx^\star) = dist(A,B).$$

Moreover, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$, defined by $d(x_{n+1}, Tx_n) = dist(A,B)$, converges to the element $x^\star$.

Remark 4.1. Since for any nonempty subset A of X, the pair (A,A) has the WP-property, we can deduce the fixed point results which was stated in the Corollaries 3.5 and 3.6, as fixed point results of Theorem 4.2.

Here, we state the relationship between proximal generalized Chatterjea contractions and generalized Chatterjea contractions.

Proposition 4.1. *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty and (A, B) has the WP-property. Assume that $T : A \to B$ is a mapping such that $T(A_0) \subseteq B_0$.*
(i) If T is a generalized Chatterjea contraction, then T is a proximal generalized Chatterjea contraction.
(ii) If T is a proximal generalized Chatterjea contraction and moreover, (B, A) has the WP- property, then $T|_{A_0}$ is a generalized Chatterjea contraction.

Proof. (i) Let T be a generalized Chatterjea contraction non-self mapping and $u, v, x, y \in A$ be such that

$$\begin{cases} d(u, Tx) = dist(A, B), \\ d(v, Ty) = dist(A, B), \end{cases} \quad \text{and} \quad \eta(r)d^*(x, Tx) \leq d(x, y).$$

Since (A, B) has the WP-property and T is generalized Chatterjea contraction,

$$d(u, v) \leq d(Tx, Ty) \leq \alpha[d^*(x, Ty) + d^*(y, Tx)],$$

which implies that T is a proximal generalized Chatterjea contraction.
(ii) Suppose that T is a proximal generalized Chatterjea contraction and $x, y \in A_0$ be such that

$$\eta(r)d^*(x, Tx) \leq d(x, y).$$

Since $T(A_0) \subseteq B_0$, there exist $u, v \in A_0$ such that

$$\begin{cases} d(u, Tx) = dist(A, B) \\ d(v, Ty) = dist(A, B). \end{cases}$$

By the fact that T is a proximal generalized Chatterjea contraction and both (A, B) and (B, A) have the WP-property, we must have

$$d(Tx, Ty) = d(u, v) \leq \alpha[d^*(x, Ty) + d^*(y, Tx)],$$

which deduces that $T|_{A_0}$ is generalized Chatterjea contraction. □

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