ON Δ-WEAK ϕ-AMENABILITY OF BANACH ALGEBRAS

Javad Laali[1] and Mohammad Fozouni[2]

Dedicated to Professor Alireza Medghalchi with appreciation and respect

Let A be a Banach algebra and $\phi \in \Delta(A) \cup \{0\}$. We introduce and study the notion of Δ-weak ϕ-amenability of Banach algebra A. It is shown that A is Δ-weak ϕ-amenable if and only if $\ker(\phi)$ *has a bounded Δ-weak approximate identity. We prove that every Δ-weak ϕ-amenable Banach algebra has a bounded Δ-weak approximate identity. Finally, we examine this notion for some algebras over locally compact groups and give a characterization of Δ-weak ϕ-amenability of the Figa-Talamanca-Herz algebras.*

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1. Introduction

Let A be a Banach algebra, $\Delta(A)$ be the character space of A, i.e., the space of all non-zero homomorphisms from A into $\mathbb{C}$ and A^* be the dual space of A consisting of all bounded linear functions from A into $\mathbb{C}$.

Throughout this paper, we assume that A is a Banach algebra such that $\Delta(A) \neq \emptyset$.

Let $\{e_\alpha\}$ be a net in a Banach algebra A. The net $\{e_\alpha\}$ is called,

(1) an *approximate identity* if, for each $a \in A$, $||ae_\alpha - a|| + ||e_\alpha a - a|| \to 0$,
(2) a *weak approximate identity* if, for each $a \in A$, $|f(ae_\alpha) - f(a)| + |f(e_\alpha a) - f(a)| \to 0$ for all $f \in A^*$,
(3) a Δ-*weak approximate identity* if, for each $a \in A$, $|\phi(e_\alpha a) - \phi(a)| \to 0$ for all $\phi \in \Delta(A)$·

The notion of weak approximate identity was originally introduced for the study of the second dual A^{**} of a Banach algebra A. For technical reasons, bounded approximate identities are of interest for mathematicians. It is proved that every Banach algebra A which has a bounded weak approximate identity, also has a bounded approximate identity and conversely [4, Proposition 33.2]. But in [9], Jones and Lahr proved that the approximate identity and Δ-weak approximate identity of a Banach algebra are different. They showed that there exists a Banach algebra A which has a bounded Δ-weak approximate identity, but it does not have any approximate identity. Indeed, if $S = \mathbb{Q}^+$ is the semigroup of positive rationales under addition,

[1,2]Faculty of Mathematical and Computer Science, Department of Mathematics, Kharazmi University, 599 Taleghani, Tehran, Iran, e-mail: laali@khu.ac.ir[1]; fozouni@khu.ac.ir[2]

they showed that the semigroup algebra $l^1(S)$ has a bounded Δ-weak approximate identity, but it does not have any bounded or unbounded approximate identity.

Definition 1.1. *Let A be a Banach algebra. A Δ-weak approximate identity for subspace $B \subseteq A$ is a net $\{a_\alpha\}$ in B such that*

$$\lim_{\alpha} |\phi(aa_\alpha) - \phi(a)| = 0 \quad (a \in B, \phi \in \Delta(A))\cdot$$

For $a \in A$ and $f \in A^*$, the linear functional $f.a$ defined as follows

$$< f.a, b >=< f, ab >= f(ab) \quad (b \in A)\cdot$$

Definition 1.2. *Let A be a Banach algebra and $\phi \in \Delta(A)$. The Banach algebra A is said to be* ϕ-amenable*, if there exists an $m \in A^{**}$ such that the following relations hold,*

(1) $m(\phi) = 1$,
(2) $m(f.a) = \phi(a)m(f) \quad (a \in A, f \in A^*)\cdot$

The concept of character amenability was first introduced by Monfared in [12]. Also, Kaniuth, Lau and Pym in [10], have investigated the concept of ϕ-amenability of Banach algebras and gave the following result.

Theorem 1.1. *Let A be a Banach algebra and $\phi \in \Delta(A)$. Then $\ker(\phi)$ has a bounded right approximate identity if and only if A is ϕ-amenable and has a bounded right approximate identity.*

Proof. See [10, Corollary 2.3]. □

In the next section of this paper, first we give the basic definition of our work, that is Δ-weak ϕ-amenability of a Banach algebra A. Then we characterize it through the existence of a bounded Δ-weak approximate identity for $\ker(\phi)$. Suppose that A is a Banach algebra and I is a closed ideal of A such that I and A/I both have bounded approximate identities. Then A has a bounded approximate identity [4, Proposition 7.1]. We give a variant of this theorem. Using this theorem, we prove that each Banach algebra A that is Δ-weak ϕ-amenable, has a bounded Δ-weak approximate identity.

In Section 3, we investigate some of the hereditary properties of Δ-weak ϕ-amenability. In section 4, we study group algebras and Figa-Talamanca Herz algebras of a locally compact group with respect to this notion and prove that when $1 < p < \infty$ and $\phi \in \Delta(A_p(G)) \cup \{0\}$, $A_p(G)$ is Δ-weak ϕ-amenable if and only if G is an amenable group.

In the final section, we just give some examples which shows the different situations that might occur for definitions and theorems in Section 2 and 3.

2. Main definition and its characterization

We begin this section with the following definition that is our main concern.

Definition 2.1. *Let A be a Banach algebra and $\phi \in \Delta(A) \cup \{0\}$. We say that A is* Δ-weak ϕ-amenable*, if there exists an $m \in A^{**}$ such that $m(\phi) = 0$ and $m(\psi.a) = \psi(a)$ for each $a \in \ker(\phi)$ and $\psi \in \Delta(A)$.*

Now, we characterize the concept of Δ-weak ϕ-amenability as follows. Recall that if A is a Banach algebra, for each $a \in A$, $\widehat{a} \in A^{**}$ is defined by $\widehat{a}(f) = f(a)$ for all $f \in A^*$.

Theorem 2.1. *Let A be a Banach algebra and $\phi \in \Delta(A) \cup \{0\}$. Then A is Δ-weak ϕ-amenable if and only if* $\ker(\phi)$ *has a bounded Δ-weak approximate identity.*

Proof. Let $\{e_\alpha\}$ be a bounded Δ-weak approximate identity for $\ker(\phi)$. So, $\{\widehat{e_\alpha}\}$ is a bounded net in A^{**}. Therefore, the Banach-Alaoglu's Theorem ([2, Theorem A.3.20]) yields $\{\widehat{e_\alpha}\}$ has a w^*-accumulation point m, i.e., there exists a subnet that we denote also by $\{\widehat{e_\alpha}\}$ such that $m = w^* - \lim_\alpha(\widehat{e_\alpha})$.

Now, we have

$$m(\phi) = \lim_\alpha \widehat{e_\alpha}(\phi) = \lim_\alpha \phi(e_\alpha) = 0,$$

and for all $\psi \in \Delta(A)$ and $a \in \ker(\phi)$,

$$m(\psi.a) = \lim_\alpha \widehat{e_\alpha}(\psi.a) = \lim_\alpha \psi.a(e_\alpha) = \lim_\alpha \psi(ae_\alpha) = \psi(a)\cdot$$

Conversely, since $m \in A^{**}$, Goldstine's Theorem ([2, Theorem A.3.29(i)]) yields there exists a net $\{e_\alpha\}$ in A such that $m = w^* - \lim_\alpha \widehat{e_\alpha}$ and $||e_\alpha|| \leq ||m||$. So, $\{e_\alpha\}$ is a bounded net such that for all $\psi \in \Delta(A)$ and $a \in \ker(\phi)$ we have

$$\begin{aligned}\psi(a) = m(\psi.a) &= \lim_\alpha \widehat{e_\alpha}(\psi.a) \\ &= \lim_\alpha \psi.a(e_\alpha) \\ &= \lim_\alpha \psi(ae_\alpha)\cdot\end{aligned}$$

Hence $\lim_\alpha |\psi(ae_\alpha) - \psi(a)| = 0$.

Note that the net $\{e_\alpha\}$ is not a subset of $\ker(\phi)$. To construct a net in $\ker(\phi)$, we have two cases $\phi = 0$ or $\phi \in \Delta(A)$. If $\phi = 0$, then the net $\{e_\alpha\}$ is a bounded Δ-weak approximate identity for $A = \ker(\phi)$.

If $\phi \in \Delta(A)$, then there exists $x_0 \in A$ such that $\phi(x_0) \neq 0$. Put $a_0 = \frac{x_0}{\phi(x_0)}$ and suppose that $a_\alpha = e_\alpha - \phi(e_\alpha)a_0$ for all α. Obviously $\{a_\alpha\} \subseteq \ker(\phi)$. On the other hand, since $m(\phi) = 0$ and $m = w^* - \lim_\alpha \widehat{e_\alpha}$, we conclude that

$$\lim_\alpha \phi(e_\alpha) = \lim_\alpha \widehat{e_\alpha}(\phi) = m(\phi) = 0\cdot$$

Hence, for each $a \in \ker(\phi)$ and $\psi \in \Delta(A)$ we have

$$\begin{aligned}\lim_\alpha |\psi(aa_\alpha) - \psi(a)| &= \lim_\alpha |\psi(ae_\alpha) - \phi(e_\alpha)\psi(aa_0) - \psi(a)| \\ &\leq \lim_\alpha |\psi(ae_\alpha) - \psi(a)| + \lim_\alpha |\phi(e_\alpha)\psi(aa_0)| \\ &= 0\cdot\end{aligned}$$

Therefore, $\lim_\alpha |\psi(aa_\alpha) - \psi(a)| = 0$ for each $a \in \ker(\phi)$ and $\psi \in \Delta(A)$, which completes the proof. □

For simplicity of notation, let b.Δ-w.a.i stand for bounded Δ-weak approximate identity and b.a.i stand for bounded approximate identity.

By Theorems 1.1 and 2.1 one can see that every ϕ-amenable Banach algebra which has a bounded right approximate identity is Δ-weak ϕ-amenable. But the converse of this assertion is not valid in general.

Remark 2.1. *Let A be a Δ-weak ϕ-amenable Banach algebra and $\{e_\alpha\}$ be a Δ-weak approximate identity of* $\ker(\phi)$*. If there exists $a_0 \in A$ with $\phi(a_0) = 1$ and $\lim_\alpha |\psi(a_0 e_\alpha) - \psi(a_0)| = 0$ for all $\psi \in \Delta(A) \setminus \{\phi\}$, then there exists $m \in A^{**}$ such that*

(1) $m(\phi) = 0$,
(2) $m(\psi.a) = \psi(a) \quad (a \in A, \psi \in \Delta(A) \setminus \{\phi\})\cdot$

By a similar argument as in the above theorem we can show the existence of m. Let $a \in A$. It is clear that $a - \phi(a)a_0 \in \ker(\phi)$. So, for all $\psi \in \Delta(A) \setminus \{\phi\}$ we have $m(\psi.(a - \phi(a)a_0)) = \psi(a - \phi(a)a_0)$. Therefore $m(\psi.a) = \psi(a)$.

Note that in part (2), it is necessary that $\psi \neq \phi$. If $\psi = \phi$ we have $m(\phi.a) = \phi(a)$. On the other hand, there exists an $a \in A$ such that $\phi(a) \neq 0$. So, we have

$$\phi(a) = m(\phi.a) = m(\phi(a)\phi) = \phi(a)m(\phi)\cdot$$

Therefore, $m(\phi) = 1$ and this is a contradiction.

The following lemma is needed in the sequel.

Lemma 2.1. *Let A be a Banach algebra such that $\phi, \psi \in \Delta(A)$ and $\phi \neq \psi$. Then there exists $a \in A$ such that $\phi(a) = 0$ and $\psi(a) = 1$.*

Proof. See the proof of [11, Theorem 3.3.14]. □

Remark 2.2. *If A is a Banach algebra with $\Delta(A) \setminus \{\phi\} \neq \emptyset$, then A is Δ-weak ϕ-amenable if and only if there exists a bounded net $\{e_\alpha\}$ in* $\ker(\phi)$ *such that $\lim_\alpha \psi(e_\alpha) = 1$ for each $\psi \in \Delta(A) \setminus \{\phi\}$ or equivalently there exists an $m \in A^{**}$ with $m(\phi) = 0$ and $m(\psi) = 1$ for all $\psi \in \Delta(A) \setminus \{\phi\}$.*

The following proposition allow us to produce Banach algebras which are Δ-weak ϕ-amenable, but they are not ϕ-amenable.

Proposition 2.1. *Let A be a Banach algebra such that $0 < |\Delta(A)| \leq 2$, i.e., $\Delta(A)$ has 1 or 2 elements. Then A is Δ-weak ϕ-amenable.*

Proof. If A only has one character, the proof is easy. Therefore, we omit it. In the second case, let $\Delta(A) = \{\phi, \psi\}$ and $\phi \neq \psi$. Hence, by Lemma 2.1 there exists an $e \in A$ with $\phi(e) = 0$ and $\psi(e) = 1$. Now, put $m = \widehat{e}$. Clearly, $m(\phi) = 0$ and $m(\psi) = 1$, so by Remark 2.2 A is Δ-weak ϕ-amenable. □

The following theorem is a useful tool in the rest of this section.

Theorem 2.2. *Let A be a Banach algebra, I be a closed two-sided ideal of A which has a b.Δ-w.a.i and the quotient Banach algebra A/I has a bounded left approximate identity (b.l.a.i). Then A has a b.Δ-w.a.i.*

Proof. Let $\{e_\alpha\}$ be a b.Δ-w.a.i for I and $\{f_\delta + I\}$ be a b.l.a.i for A/I. Suppose that $F = \{a_1, ...a_m\}$ is a finite subset of A and n is a positive integer. Let M be an upper bound for $\{||e_\alpha||\}$. For $\lambda = (F, n)$, there exists f_{δ_λ} such that

$$||f_{\delta_\lambda} a_i - a_i + I|| < \frac{1}{2(1+M)n} \qquad (i = 1, 2, 3, ..., m)\cdot$$

Therefore, there exists $y_i \in I$ such that

$$||f_{\delta_\lambda} a_i - a_i + y_i|| < \frac{1}{2(1+M)n} \qquad (i = 1, 2, 3, ..., m)\cdot$$

Let $\psi \in \Delta(A)$. Since $\{e_\alpha\}$ is a b.Δ-w.a.i, for each y_i with $i \in \{1, 2, 3, ..., m\}$ which satisfy the above relation, there exists $e_{\alpha_\lambda} \in \{e_\alpha\}$ such that

$$|\psi(e_{\alpha_\lambda} y_i) - \psi(y_i)| < \frac{1}{2n} \quad (i = 1, 2, 3, ..., m)\cdot$$

Now, for each $i \in \{1, 2, 3, ..., m\}$ we have

$$\begin{aligned} |\psi((e_{\alpha_\lambda} + f_{\delta_\lambda} - e_{\alpha_\lambda} f_{\delta_\lambda}) a_i) - \psi(a_i)| &\le |\psi(f_{\delta_\lambda} a_i - a_i + y_i)| \\ &+ |\psi(e_{\alpha_\lambda} y_i) - \psi(y_i)| \\ &+ |\psi(e_{\alpha_\lambda} a_i - e_{\alpha_\lambda} f_{\delta_\lambda} a_i - e_{\alpha_\lambda} y_i)| \\ &\le ||f_{\delta_\lambda} a_i - a_i + y_i|| \\ &+ \frac{1}{2n} + M||a_i - f_{\delta_\lambda} a_i - y_i|| \\ &< \frac{1}{n}\cdot \end{aligned}$$

Therefore, $\{e_{\alpha_\lambda} + f_{\delta_\lambda} - e_{\alpha_\lambda} f_{\delta_\lambda}\}_{\lambda \in \Lambda}$ is a Δ-w.a.i for A, where $\Lambda = \{(F, n) : F \subseteq A \text{ is finite}, n \in \mathbb{N}\}$ is a directed set with $(F_1, n_1) \le (F_2, n_2)$ if $F_1 \subseteq F_2$ and $n_1 \le n_2$.

Now, we show that there exists a b.Δ-w.a.i for A. Since $\{f_\delta + I\}$ is bounded, there exists a positive integer K such that $||f_\delta + I|| < K$ for each δ. So, there exists $y_\delta \in I$ such that $||f_\delta + I|| < ||f_\delta + y_\delta|| < K$. Put $f'_\delta = f_\delta + y_\delta$. Hence, $\{f'_\delta + I\}$ is a bounded approximate identity for A/I which $\{f'_\delta\}$ is bounded. Now, we have

$$||e_{\alpha_\lambda} + f'_{\delta_\lambda} - e_{\alpha_\lambda} f'_{\delta_\lambda}|| \le ||e_{\alpha_\lambda}|| + ||f'_{\delta_\lambda}|| + ||e_{\alpha_\lambda}|| ||f'_{\delta_\lambda}|| < M + K + KM\cdot$$

Therefore, A has a b.Δ-w.a.i. □

It is straightforward to see that for every closed two-sided ideal I with codimension one of a Banach algebra A, the quotient Banach algebra A/I has a bounded approximate identity. So, we have the following corollary.

Corollary 2.1. *Let A be a Banach algebra and $\phi \in \Delta(A) \cup \{0\}$. If A is Δ-weak ϕ-amenable, then A has a b.Δ-w.a.i.*

Proof. Since A is Δ-weak ϕ-amenable, by Theorem 2.1, $\ker(\phi)$ has a b.Δ-w.a.i. Also, $A/\ker(\phi)$ has a b.a.i, because the codimension of $\ker(\phi)$ is one. Then, by Theorem 2.2, A has a b.Δ-w.a.i. □

Corollary 2.2. *Let A be a Banach algebra and $\phi \in \Delta(A)$. If A is Δ-weak ϕ-amenable, then A is Δ-weak 0-amenable.*

The converse of the above corollary is not valid in general (see Example 5.6).

Note that there exist Banach algebras which do not have any b.Δ-w.a.i. As an example, let G be a locally compact group and $1 < p < \infty$. Consider $S_p(G) = L^1(G) \cap L^p(G)$ with the norm defined by $||f||_{S_p(G)} = \max\{||f||_1, ||f||_p\}$ which is a Segal algebra (see [18] for a full discussion on Segal algebras). Now, by [8, Remark 2], if G is an infinite abelian compact group, $S_p(G)$ has no b.Δ-w.a.i.

There exists a Δ-weak version of an identity in a Banach algebra A.

Definition 2.2. *Let A be a Banach algebra. We say that $e \in A$ is a Δ-weak identity for A if for each $\phi \in \Delta(A)$, $\phi(e) = 1$ or equivalently*

$$\phi(ea) = \phi(a) \quad (a \in A, \phi \in \Delta(A))\cdot$$

It is obvious that the identity of a Banach algebra A is a Δ-weak identity of A, but the converse is not valid in general.

The following theorem gives a necessary condition for Δ-weak ϕ-amenability of finite dimensional Banach algebras.

Theorem 2.3. *Let A be a finite dimensional Banach algebra. If A is Δ-weak ϕ-amenable, then it has a Δ-weak identity.*

Proof. In view of Corollary 2.1, A has a b.Δ-w.a.i, say $\{e_\alpha\}$. By the Heine-Borel's Theorem ([15, Theorem 2.38]), we know that every closed and bounded subset of a finite dimensional normed linear space is compact.

So, $\{e_\alpha\}$ is relatively compact, i.e., its closure, $\overline{\{e_\alpha\}}$ is compact, because it is closed and bounded. Therefore, there exists $e \in A$ and a convergent subnet that we denote also by $\{e_\alpha\}$ such that converges to e. Now, for each $\psi \in \Delta(A)$ and $a \in A$, we have

$$\psi(ea) = \lim_\alpha \psi(e_\alpha a) = \psi(a)\cdot$$

Therefore, e is a Δ-weak identity for A. □

3. Hereditary properties

In this section, we give some of the hereditary properties of Δ-weak ϕ-amenability.

Theorem 3.1. *Let A and B be Banach algebras, $\phi \in \Delta(B)$ and $h : A \to B$ be a dense range continuous homomorphism. If A is Δ-weak $\phi \circ h$-amenable, then B is Δ-weak ϕ-amenable.*

Proof. Let A be Δ-weak $\phi \circ h$-amenable. So, there exists $m \in A^{**}$ such that, $m(\phi \circ h) = 0$ and $m(\psi.a) = \psi(a)$ for all $a \in \ker(\phi \circ h)$ and $\psi \in \Delta(A)$.

Define $n \in B^{**}$ as follows

$$n(g) = m(g \circ h) \quad (g \in B^*)\cdot$$

So, $n(\phi) = m(\phi \circ h) = 0$. For each $b \in \ker(\phi)$, there exists a sequence $\{e_n\}$ in A such that $\lim_n h(e_n) = b$. Put $a_n = e_n - \phi \circ h(e_n)a_0$ where $\phi \circ h(a_0) = 1$. It is obvious that $a_n \in \ker(\phi \circ h)$ for each n and $\lim_n h(a_n) = b$. Also, for each $\psi' \in \Delta(B)$, $(\psi' \circ h).a_n \to (\psi'.b) \circ h$ in A^*, since

$$\begin{aligned} ||(\psi' \circ h).a_n - (\psi'.b) \circ h|| &= \sup_{||a|| \le 1} ||\psi'(h(a_n a)) - \psi'(bh(a))|| \\ &\le \sup_{||a|| \le 1} ||h(a_n)h(a) - bh(a)|| \\ &\le ||h(a_n) - b||||h||\cdot \end{aligned}$$

Therefore, for all $\psi' \in \Delta(B)$ we have

$$\begin{aligned} n(\psi'.b) = m((\psi'.b) \circ h) &= \lim_n m((\psi' \circ h).a_n) \\ &= \lim_n \psi' \circ h(a_n) \\ &= \lim_n \psi'(h(a_n)) \\ &= \psi'(b)\cdot \end{aligned}$$

So, B is Δ-weak ϕ-amenable. □

Let I be a closed ideal of a Banach algebra A. If I has an approximate identity, then every $\phi \in \Delta(I)$ extends to some $\widetilde{\phi} \in \Delta(A)$. To see this let $\{e_\alpha\}$ be an approximate identity of I and $u \in I$ be an element with $\phi(u) = 1$. If $a \in A$ and $b \in \ker(\phi)$, then we have

$$\phi(ab) = \lim_{\alpha} \phi(ae_\alpha b) = 0\cdot$$

Therefore, $ab \in \ker(\phi)$ and this shows that $\ker(\phi)$ is a left ideal in A. Now, Define $\widetilde{\phi}$: $A \to \mathbb{C}$ by $\widetilde{\phi}(a) = \phi(au)$ for all $a \in A$. Since for $a, b \in A$, $bu - ubu \in \ker(\phi)$ therefore, $abu - aubu = a(bu - ubu) \in \ker(\phi)$. So, we conclude that $\phi(abu) = \phi(au)\phi(bu)$. Hence

$$\widetilde{\phi}(ab) = \phi(abu) = \phi(au)\phi(bu) = \widetilde{\phi}(a)\widetilde{\phi}(b) \quad (a, b \in A)\cdot$$

Proposition 3.1. *Let A be a Banach algebra, I be a closed ideal of A which has a bounded approximate identity and $\phi \in \Delta(A)$ with $I \nsubseteq \ker(\phi)$. If A is Δ-weak ϕ-amenable, then I is Δ-weak $\phi_{|I}$-amenable.*

Proof. Let $\{a_\beta\}$ be a bounded approximate identity for I and $\{e_\alpha\}$ be a b.Δ-w.a.i for $\ker(\phi)$. It is clear that $\lim_\beta \psi(a_\beta) = 1$ for all $\psi \in \Delta(I)$.

Put $c_{(\alpha,\beta)} = e_\alpha a_\beta$ for all α, β. Then $\{c_{(\alpha,\beta)}\}_{(\alpha,\beta)}$ is a bounded net in I. Now, for each $a \in \ker(\phi_{|I})$ and $\psi \in \Delta(I)$ we have

$$\begin{aligned}\lim_{(\alpha,\beta)} \psi(ac_{(\alpha,\beta)}) &= \lim_{(\alpha,\beta)} \psi(ae_\alpha a_\beta)\\ &= (\lim_{(\alpha,\beta)} \psi(ae_\alpha))(\lim_{(\alpha,\beta)} \psi(a_\beta))\\ &= (\lim_{(\alpha,\beta)} \widetilde{\psi}(ae_\alpha))(\lim_{(\alpha,\beta)} \psi(a_\beta))\\ &= \widetilde{\psi}(a) = \psi(a)\cdot\end{aligned}$$

Therefore, I is Δ-weak $\phi_{|I}$-amenable by Theorem 2.1. □

For each Banach algebra A, we can extend each $\phi \in \Delta(A)$ uniquely to a character ϕ of A^{**} defined by $\widehat{\phi}(a^{**}) = a^{**}(\phi)$ for all $a^{**} \in A^{**}$. So, we have the following result.

Proposition 3.2. *Let A be a Banach algebra and $\phi \in \Delta(A)$. If A^{**} is Δ-weak $\widehat{\phi}$-amenable, then A is Δ-weak ϕ-amenable.*

Proof. Let A^{**} be Δ-weak $\widehat{\phi}$-amenable. So, there exists $m^{**} \in A^{****}$ which satisfies the following relations,

(1) $m^{**}(\widehat{\phi}) = 0$,
(2) $m^{**}(\Psi.F) = \Psi(F) \quad (\Psi \in \Delta(A^{**}), F \in \ker(\widehat{\phi}))\cdot$

Put $m(f) = m^{**}(\widehat{f})$ for all $f \in A^*$. Therefore, $m(\phi) = m^{**}(\widehat{\phi}) = 0$ and for each $a \in \ker(\phi) \subseteq \ker(\widehat{\phi})$ we have

$$m(\psi.a) = m^{**}(\widehat{\psi.a}) = m^{**}(\widehat{\psi}.\widehat{a}) = \widehat{\psi}(\widehat{a}) = \widehat{a}(\psi) = \psi(a) \quad (\psi \in \Delta(A))\cdot$$

Therefore, A is Δ-weak ϕ-amenable. □

4. **Some results on algebras over locally compact groups**

Let G be a locally compact group. For $1 < p < \infty$ let $A_p(G)$ denotes the subspace of $C_0(G)$ consisting of functions of the form $u = \sum_{i=1}^{\infty} f_i * \widetilde{g_i}$ where $f_i \in L^p(G)$, $g_i \in L^q(G)$, $1/p + 1/q = 1$, $\sum_{i=1}^{\infty} ||f_i||_p ||g_i||_q < \infty$ and $\widetilde{f}(x) = \overline{f(x^{-1})}$ for all $x \in G$. $A_p(G)$ is called the Figa-Talamanca-Herz algebra and with the pointwise operation and the following norm is a Banach algebra,

$$||u||_{A_p(G)} = \inf\{\sum_{i=1}^{\infty} ||f_i||_p ||g_i||_q : u = \sum_{i=1}^{\infty} f_i * \widetilde{g_i}\}\cdot$$

It is obvious that for each $u \in A_p(G)$, $||u|| \leq ||u||_{A_p(G)}$ where $||u||$ is the norm of u in $C_0(G)$. Also, we know that $\Delta(A_p(G)) = G$, i.e., each character of $A_p(G)$ is an evaluation function at some $x \in G$ [5, Theorem 3].

The dual of the Banach algebra $A_p(G)$ is the Banach space $PM_p(G)$ consisting of all limits of convolution operators associated to bounded measures [3, Chapter 4].

The group G is said to be amenable if, there exists an $m \in L^\infty(G)^*$ such that $m \geq 0$, $m(1) = 1$ and $m(L_x f) = m(f)$ for each $x \in G$ and $f \in L^\infty(G)$ where $L_x f(y) = f(x^{-1}y)$ [17, Definition 4.2].

First we give the following lemma which is a generalization of Leptin-Herz Theorem ([17, Theorem 10.4]).

Lemma 4.1. *Let G be a locally compact group and $1 < p < \infty$. Then $A_p(G)$ has a b.Δ-w.a.i if and only if G is amenable.*

Proof. Let $\{e_\alpha\}$ be a b.Δ-w.a.i for $A_p(G)$ and $e \in A_p(G)^{**}$ be a w^*-cluster point of $\{e_\alpha\}$.

So, for each $\phi \in \Delta(A_p(G)) = G$, we have

$$< e, \phi >= \lim_{\alpha} \phi(e_\alpha) = 1\cdot$$

Therefore, by [19, Proposition 2.8] G is weakly closed in $PM_p(G) = A_p(G)^*$. Now, by [1, Corollary 2.8] we conclude that G is an amenable group. □

For a locally compact group G, let $L^1(G)$ be the group algebra of G endowed with the norm $||.||_1$ and the convolution product as defined in [6]. By [7, Theorem 23.7] we know that

$$\Delta(L^1(G)) = \{\phi_\rho; \rho \in \widehat{G}\},$$

where $\widehat{G}$ is the space of all continuous homomorphisms from G into the circle group $\mathbb{T}$ and ϕ_ρ defined by

$$\phi_\rho(h) = \int_G \overline{\rho(x)} h(x) dx \quad (h \in L^1(G))\cdot$$

Theorem 4.1. *Let G be a locally compact group.*

(1) *If G is an amenable group, then $L^1(G)$ is Δ-weak ϕ-amenable for each $\phi \in \Delta(L^1(G)) \cup \{0\}$.*
(2) *For $1 < p < \infty$ and $\phi \in \Delta(A_p(G)) \cup \{0\}$, $A_p(G)$ is Δ-weak ϕ-amenable if and only if G is an amenable group.*

Proof. (1): It follows from [12, Corollary 2.4] that $L^1(G)$ is ϕ-amenable for all $\phi \in \Delta(L^1(G))$. Also, we know that the group algebra has a bounded approximate identity and this completes the proof of (1) by using Theorem 1.1.

(2): If G is an amenable group, then by Leptin-Herz's Theorem we know that $A_p(G)$ has a bounded approximate identity. On the other hand, by [12, Corollary 2.4], $A_p(G)$ is ϕ-amenable for each $\phi \in \Delta(A_p(G))$. So, the result follows from Theorem 1.1.

Conversely, let $A_p(G)$ be Δ-weak ϕ-amenable. If $\phi = 0$ the result follows from Lemma 4.1. If $\phi \in \Delta(A_p(G))$, by Corollary 2.1 we know that $A_p(G)$ has a b.Δ.w.a.i. So, the result follows from Lemma 4.1. □

5. **Examples**

In this section, we only give some instructive examples.

The following example shows that the Δ-weak 0-amenability and 0-amenability are different.

Example 5.1. *Let $S = \mathbb{Q}^+$ be the semigroup of positive rational numbers under addition. So, $A = l^1(\mathbb{Q}^+)$ is Δ-weak 0-amenable, but it is not 0-amenable. The reason is that A has a b.Δ-w.a.i, but it does not have any approximate identity* [9].

The following three examples give Banach algebras which are Δ-weak ϕ-amenable, but they are not ϕ-amenable.

Example 5.2. *Let X be a Banach space and take $\phi \in X^* \setminus \{0\}$ with $||\phi|| \leq 1$. Define a product on X by $ab = \phi(a)b$ for all $a, b \in X$. With this product X is a Banach algebra which we denote it by $A_\phi(X)$. It is clear that $\Delta(A_\phi(X)) = \{\phi\}$ and $A_\phi(X)$ is ϕ-amenable if and only if $dim(X) = 1$* [16, Example 2.4].

Thus, if we take a Banach space X with $dim(X) > 1$ and a non-injective $\phi \in X^*$, then $A_\phi(X)$ is not ϕ-amenable, but it is Δ-weak ϕ-amenable by Proposition 2.1.

On the other hand, let $x_0 \in X$ be such that $\phi(x_0) = 1$. Then x_0 is a Δ-weak identity for $A_\phi(X)$, but it is not an identity. Because, for $0 \neq a \in ker(\phi)$ we have $ax_0 = \phi(a)x_0 = 0$. Therefore, $ax_0 \neq a$. So, x_0 is not an identity. Moreover, it is clear that the Δ-weak identity of the Banach algebra $A_\phi(X)$ is not unique, since each element $a_0 \in X$ such that $\phi(a_0) = 1$, is a Δ-weak identity for $A_\phi(X)$.

Let A and B be Banach algebras with $\Delta(B) \neq \emptyset$ and $\theta \in \Delta(B)$. The θ-Lau product $A \times_\theta B$ is defined as the Cartesian product $A \times B$ with the following multiplication,

$$(a, b)(a_1, b_1) = (aa_1 + \theta(b)a_1 + \theta(b_1)a, bb_1) \quad (a, a_1 \in A, b, b_1 \in B)\cdot$$

With the l^1-norm and the above multiplication, $A \times_\theta B$ is a Banach algebra; see [13].

Example 5.3. *Let $A = B = A_\phi(X)$ be the Banach algebra defined in the above example and $dim(X) > 1$. Consider the ϕ-Lau product $A_\phi(X) \times_\phi A_\phi(X)$. By* [13, Proposition 2.8]*, we know that $|\Delta(A_\phi(X) \times_\phi A_\phi(X))| = 2$. Let $\Delta(A_\phi(X) \times_\phi A_\phi(X)) = \{\Theta_1, \Theta_2\}$. So, by Proposition 2.1, $A_\phi(X) \times_\phi A_\phi(X)$ is Δ-weak Θ_1-amenable and Δ-weak Θ_2-amenable. On the other hand, $A_\phi(X)$ is not ϕ-amenable. Hence, by* [14, Lemma 6.8 (iii)] *there exists a character Θ_i, $i = 1$ or 2 of $A_\phi(X) \times_\phi A_\phi(X)$ such that $A_\phi(X) \times_\phi A_\phi(X)$ is not Θ_i-amenable.*

Example 5.4. *Let $n \geq 2$ be an integer number and let A be the Banach algebra of all upper-triangular $n \times n$ matrix over $\mathbb{C}$. We have $\Delta(A) = \{\phi_1, \phi_2, \phi_3, ..., \phi_n\}$ where*

$$\phi_k([a_{ij}]) = a_{kk} \quad (k = 1, 2, 3, ..., n)\cdot$$

Then for each ϕ_k, A is Δ-weak ϕ_k-amenable. To see this, let $e_0 = [a_{ij}]$ be an element of A such that

$$a_{ij} = \begin{cases} 1 & i = j, i \neq k \\ 0 & i = j = k \\ 0 & i \neq j \end{cases} \cdot$$

Obviously, e_0 is in $\ker(\phi_k)$. But $\phi_i(e_0) = 1$ for all $i \neq k$. So, the result follows by using Theorem 2.1.

Also, A is not ϕ_k-amenable for each $k \geq 2$. Because, $\ker(\phi_k)$ does not have a right identity. Therefore, by [10, Proposition 2.2] *A is not ϕ_k-amenable.*

Some Banach algebras satisfy both concepts of ϕ-amenability and Δ-weak ϕ-amenability.

Example 5.5. *If A is a C^*-algebra, then A is ϕ-amenable and Δ-weak ϕ-amenable, because each C^*-algebra and their closed ideals have bounded approximate identity* [2, Theorem 3.2.21].

There exists a Banach algebra that is neither ϕ-amenable nor Δ-weak ϕ-amenable as the next example shows. Also, the following example shows that the converse of Corollary 2.2 is not valid in general.

Example 5.6. *Let $A = C^1[0, 1]$ be the Banach algebra consisting of all continuous functions on $[0, 1]$ with continuous derivation and norm $||f||_1 = ||f||_\infty + ||f'||_\infty$. We know that*

$$\Delta(A) = \{\phi_t : \phi_t(f) = f(t) \text{ for each } t \in [0, 1]\}\cdot$$

By [10, Example 2.5(I)], *A is not ϕ_t-amenable for any $t \in [0, 1]$. Moreover, there does not exist $t_0 \in [0, 1]$ such that A is Δ-weak ϕ_{t_0}-amenable. To see this, let $\{f_n\}$ be a b.Δ-w.a.i for $\ker(\phi_{t_0})$. So, it has the following properties*

(1) $\lim_n f_n(t) = 1 \quad (t \in [0, 1] \setminus \{t_0\})$,
(2) $\{||f_n'||_\infty\}$ *is bounded*$\cdot$

Hence, there exists a non-negative constant M with $||f_n'||_\infty = \sup_{t\in[0,1]} |f_n'(t)| < M$ for all $n \in \mathbb{N}$. Hence, for positive integer n_0 we have

$$\lim_{t\to t_0} |\frac{f_{n_0}(t) - f_{n_0}(t_0)}{t - t_0}| = |f_{n_0}'(t_0)| < M\cdot$$

Therefore, there exists $\epsilon > 0$ such that for each $t \in N(t_0, \epsilon) = \{t : 0 < |t - t_0| < \epsilon\}$ we have

$$|f_{n_0}(t) - f_{n_0}(t_0)| < M|t - t_0|\cdot$$

But the above relation is not valid in general, because the right hand side of the inequality tends to zero as $t \to t_0$, but the left hand side does not.

Hence, A is not Δ-weak ϕ_t-amenable for each $t \in [0, 1]$.

Also, this Banach algebra is Δ-weak 0-amenable, because the sequence $\{\frac{n-t^n}{n}\}$ is a bounded Δ-weak approximate identity for A.

The converse of Theorem 3.1 does not hold in general as the following example shows.

Example 5.7. *Let $A = C^1[0,1]$, $B = C[0,1]$ and $h : A \hookrightarrow B$ be the inclusion homomorphism. It is clear that A is dense in B. By Examples 5.5 and 5.6 for each $t \in [0,1]$, B is Δ-weak ϕ_t-amenable, but A is not Δ-weak ϕ_t-amenable.*

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