GENERAL QUARTIC-CUBIC-QUADRATIC FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN NORMED SPACES

M. Eshaghi Gordji[1], H. Khodaei[2], R. Khodabakhsh[3]

The aim of this paper is to find the general solution of a mixed type quartic, cubic and quadratic functional equation

$$f(x+ky)+f(x-ky)=k^2f(x+y)+k^2f(x-y)+2(1-k^2)f(x) + \frac{k^2(k^2-1)}{6}(f(2y)+2f(-y)-6f(y))$$

($k\in\mathbb{Z}-\{0,\pm1\}$) in the class of functions between real vector spaces and to obtain the generalized Hyers-Ulam stability problem for the equation in non-Archimedean spaces.

Keywords: Quartic, cubic and quadratic functions; Non-Archimedean spaces; p–adic field; Stability

MSC2000: 39B82, 39B52.

1. Introduction

In 1897, Hensel [9] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications [13, 20, 22, 23].

A non-Archimedean field is a field K equipped with a function (valuation) $|\ .\ |$ from K into $[0,\infty)$ such that $|r|=0$ if and only if $r=0$, $|rs|=|r||s|$, and $|r+s|\le\max\{|r|,|s|\}$ for all $r,s\in K$. Clearly $|1|=|-1|=1$ and $|n|\le 1$ for all $n\in N$. An example of a non-Archimedean valuation is the function $|\ .\ |$ taking everything but 0 into 1 and $|0|=0$. This valuation is called trivial.

Definition 1.1. *Let X be a vector space over a scalar field K with a non–Archimedean non-trivial valuation $|\ .\ |$. A function $\|\ .\ \|:X\to\mathbb{R}$ is a non–Archimedean norm (valuation) if it satisfies the following conditions:*

(NA1) $\|x\|=0$ *if and only if* $x=0$;

(NA2) $\|rx\|=|r|\|x\|$ *for all* $r\in K$ *and* $x\in X$;

(NA3) $\|x+y\|\le\max\{\|x\|,\|y\|\}$ *for all* $x,y\in X$ *(the strong triangle inequality).*

Then $(X,\|\ .\ \|)$ *is called a non–Archimedean space.*

[1]Associate Professor, Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran, e-mail: madjid.eshaghi@gmail.com

[2]Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran, khodaei.hamid.math@gmail.com

[3]Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran, raziehkhodabakhsh@gmail.com

Remark 1.1. *Thanks to the inequality*

$$\|x_m - x_l\| \leq \max\{\|x_{\jmath+1} - x_\jmath\| : l \leq \jmath \leq m-1\} \quad (m > l)$$

a sequence $\{x_m\}$ is Cauchy if and only if $\{x_{m+1} - x_m\}$ converges to zero in a non–Archimedean space. By a complete non–Archimedean space we mean one in which every Cauchy sequence is convergent.

The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: "for $x, y > 0$, there exists $n \in \mathbb{N}$ such that $x < ny$."

Example 1.1. *Let p be a prime number. For any nonzero rational number $x = \frac{a}{b}p^r$ such that a and b are coprime to the prime number p, define the p-adic absolute value $|x|_p := p^{-r}$. Then $|\ .\ |_p$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\ .\ |_p$ is denoted by $\mathbb{Q}_p$ and is called the p-adic number field.*

Note that if $p > 3$, then $|2^n| = 1$ in for each integer n.
The concept of stability of a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. In 1940, Ulam [21] posed the first stability problem. In the next year, Hyers [10] gave an affirmative answer to the question of Ulam. Hyers theorem was generalized by Aoki [2] for additive mappings and by Rassias [16] for linear mappings by considering an unbounded Cauchy difference. The concept of the Hyers-Ulam-Rassias stability originated from Rassias paper [16] for the stability of functional equations (see also [8, 17, 18, 19]).

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1}$$

is related to a symmetric bi–additive function [1, 12]. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1) is said to be a quadratic function. It is well known that a function f between real vector spaces X and Y is quadratic if and only if there exists a unique symmetric bi-additive function B_1 such that $f(x) = B_1(x, x)$ for all $x \in X$. The bi-additive function B_1 is given by

$$B_1(x, y) = \frac{1}{4}(f(x+y) - f(x-y))$$

for all $x, y \in X$. In the paper [5], Czerwik proved the Hyers–Ulam–Rassias stability of the equation (1).

Jun and Kim [11] introduced the following functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{2}$$

and they established the general solution and the generalized Hyers–Ulam–Rassias stability for functional equation (2). They proved that a function f between real vector spaces X and Y is a solution of (2) if and only if there exits a unique function $C : X \times X \times X \longrightarrow Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables. The function C is given by

$$C(x, y, z) = \frac{1}{24}(f(x+y+z) + f(x-y-z) - f(x+y-z) - f(x-y+z))$$

for all $x, y, z \in X$. It is easy to see that the function $f(x) = cx^3$ satisfies functional equation (2), so it is natural to call (2) the cubic functional equation and every solution of the cubic functional equation (2) is said to be a cubic function.

Lee et. al. [14] considered the following functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y) \tag{3}$$

In fact, they proved that a function f between two real vector spaces X and Y is a solution of (3) if and only if there exists a unique symmetric bi-quadratic function $B_2 : X \times X \longrightarrow Y$ such that $f(x) = B_2(x, x)$ for all x. The bi-quadratic function B_2 is given by

$$B_2(x,y)=\frac{1}{12}(f(x+y)+f(x-y)-2f(x)-2f(y))$$

for all $x, y \in X$. It is easy to show that the function $f(x) = dx^4$ satisfies the functional equation (3), which is called the quartic functional equation (see also [4]).

In 2007, Moslehian and Rassias [15] proved the generalized Hyers–Ulam stability of the Cauchy functional equation and the quadratic functional equation in non–Archimedean normed spaces.

Eshaghi Gordji and Khodaei [6], have obtained the generalized Hyers–Ulam–Rassias stability for a mixed type of cubic, quadratic and additive functional equation. In addition the generalized Hyers–Ulam–Rassias stability for cubic and quartic functional equation in non-Archimedean space has been investigated by Eshaghi Gordji and Bavand Savadkouhi [7].

In this paper, we deal with the following mixed type quartic, cubic and quadratic functional equation for fixed integers $k \neq \pm 1$,

$$\begin{aligned} f(x+ky)+f(x-ky) &= k^2f(x+y)+k^2f(x-y)+2(1-k^2)f(x) \\ &\quad + \frac{k^2(k^2-1)}{6}(f(2y)+2f(-y)-6f(y)) \end{aligned} \tag{4}$$

It is easy to see that the function $f(x) = dx^4+cx^3+bx^2$ is a solution of the functional equation (4).

The main purpose of this paper is to establish the general solution of Eq. (4) and investigate the generalized Hyers-Ulam stability for Eq. (4) in non-Archimedean spaces.

2. Functional equations deriving from quartic, cubic and quadratic functions

We here present the general solutions of (4).

Theorem 2.1. *Let both X and Y be real vector spaces. A function $f : X \to Y$ satisfies (4) for all $x, y \in X$ if and only if there exist a unique symmetric bi-quadratic function $B_2 : X \times X \longrightarrow Y$, a unique function $C : X \times X \times X \longrightarrow Y$ and a unique symmetric bi-additive function $B_1 : X \times X \longrightarrow Y$ such that*

$$f(x) = B_2(x, x) + C(x, x, x) + B_1(x, x)$$

for all $x \in X$, and that C is symmetric for each fixed one variable and is additive for fixed two variables.

Proof. Let f satisfies (4). We decompose f into the even part and odd part by setting

$$f_e(x) = \frac{1}{2}(f(x) + f(-x)), \qquad f_o(x) = \frac{1}{2}(f(x) - f(-x))$$

for all $x \in X$. It is clear that $f(x) = f_e(x) + f_o(x)$ for all $x \in X$. It is easy to show that the functions f_e and f_o satisfy (4). Now, let $g, h : X \to Y$ be functions defined by

$$g(x) = f_e(2x) - 16 f_e(x), \qquad h(x) = f_e(2x) - 4 f_e(x)$$

for all $x \in X$. We show that the functions g and h are quadratic and quartic, respectively.

Interchanging x with y in (4) and then using $f_e(-x) = f_e(x)$, we have

$$\begin{aligned} f_e(kx+y) + f_e(kx-y) &= k^2 f_e(x+y) + k^2 f_e(x-y) \\ &\quad + \frac{k^2(k^2-1)}{6}(f_e(2x) - 4f_e(x)) + 2(1-k^2) f_e(y) \end{aligned} \tag{5}$$

for all $x, y \in X$. Letting $x = y = 0$ in (5), we have $f_e(0) = 0$. Putting $y = x + y$ in (5), gives

$$\begin{aligned} f_e((k+1)x+y) + f_e((k-1)x-y) &= k^2 f_e(2x+y) + k^2 f_e(-y) \\ &\quad + \frac{k^2(k^2-1)}{6}(f_e(2x) - 4f_e(x)) + 2(1-k^2) f_e(x+y) \end{aligned} \tag{6}$$

for all $x, y \in X$. Replacing y by $-y$ in (6), we obtain

$$\begin{aligned} f_e((k+1)x-y) + f_e((k-1)x+y) &= k^2 f_e(2x-y) + k^2 f_e(y) \\ &\quad + \frac{k^2(k^2-1)}{6}(f_e(2x) - 4f_e(x)) + 2(1-k^2) f_e(x-y) \end{aligned} \tag{7}$$

for all $x, y \in X$. Adding (6) to (7), we get

$$\begin{aligned} &f_e((k+1)x+y) + f_e((k+1)x-y) + f_e((k-1)x+y) + f_e((k-1)x-y) \\ &= k^2(f_e(2x+y) + f_e(2x-y)) + \frac{2k^2(k^2-1)}{6}(f_e(2x) - 4f_e(x)) \\ &\quad + 2(1-k^2)(f_e(x+y) + f_e(x-y)) + 2k^2 f_e(y) \end{aligned} \tag{8}$$

for all $x, y \in X$. From the substitution $y = kx + y$ in (5), we have

$$\begin{aligned} f_e(2kx+y) + f_e(y) &= k^2 f_e((k+1)x+y) + k^2 f_e((k-1)x+y) \\ &\quad + \frac{k^2(k^2-1)}{6}(f_e(2x) - 4f_e(x)) + 2(1-k^2) f_e(kx+y) \end{aligned} \tag{9}$$

for all $x, y \in X$. Replacing y by $-y$ in (9), we get

$$\begin{aligned} f_e(2kx-y) + f_e(-y) &= k^2 f_e((k+1)x-y) + k^2 f_e((k-1)x-y) \\ &\quad + \frac{k^2(k^2-1)}{6}(f_e(2x) - 4f_e(x)) + 2(1-k^2) f_e(kx-y) \end{aligned} \tag{10}$$

for all $x, y \in X$. Adding (9) to (10), we obtain

$$\begin{aligned} f_e(2kx+y)+f_e(2kx-y) &= k^2(f_e((k+1)x+y)+f_e((k+1)x-y) \\ &\quad + f_e((k-1)x+y)+f_e((k-1)x-y)) \\ &\quad + 2(1-k^2)(f_e(kx+y)+f_e(kx-y)) \\ &\quad + \frac{2k^2(k^2-1)}{6}(f_e(2x)-4f_e(x))-2f_e(y) \end{aligned} \tag{11}$$

for all $x, y \in X$. It follows from (11) using (5) and (8) that

$$\begin{aligned} &f_e(2kx+y)+f_e(2kx-y) \\ &\quad = k^2[k^2(f_e(2x+y)+f_e(2x-y))+\frac{2k^2(k^2-1)}{6}(f_e(2x)-4f_e(x)) \\ &\qquad + 2(1-k^2)(f_e(x+y)+f_e(x-y))+2k^2f_e(y)] \\ &\quad + 2(1-k^2)[k^2f_e(x+y)+k^2f_e(x-y) \\ &\quad + \frac{k^2(k^2-1)}{6}(f_e(2x)-4f_e(x))+2(1-k^2)f_e(y)] \\ &\quad + \frac{2k^2(k^2-1)}{6}(f_e(2x)-4f_e(x))-2f_e(y) \end{aligned} \tag{12}$$

for all $x, y \in X$. If we replace x by $2x$ in (5), we get that

$$\begin{aligned} f_e(2kx+y)+f_e(2kx-y) &= k^2f_e(2x+y)+k^2f_e(2x-y) \\ &\quad + \frac{k^2(k^2-1)}{6}(f_e(4x)-4f_e(2x))+2(1-k^2)f_e(y) \end{aligned} \tag{13}$$

for all $x, y \in X$. It follows from (12) and (13) that

$$\begin{aligned} &k^2[k^2(f_e(2x+y)+f_e(2x-y))+\frac{2k^2(k^2-1)}{6}(f_e(2x)-4f_e(x)) \\ &\quad + 2(1-k^2)(f_e(x+y)+f_e(x-y))+2k^2f_e(y)] \\ &+ 2(1-k^2)[k^2f_e(x+y)+k^2f_e(x-y) \\ &+ \frac{k^2(k^2-1)}{6}(f_e(2x)-4f_e(x))+2(1-k^2)f_e(y)] \\ &+ \frac{2k^2(k^2-1)}{6}(f_e(2x)-4f_e(x))-2f_e(y) \\ &\quad = k^2f_e(2x+y)+k^2f_e(2x-y)+\frac{k^2(k^2-1)}{6}(f_e(4x)-4f_e(2x)) \\ &\qquad + 2(1-k^2)f_e(y) \end{aligned} \tag{14}$$

for all $x, y \in X$. Also, putting $y = 0$ in (5), we get

$$f_e(kx) = k^2f_e(x)+\frac{k^2(k^2-1)}{12}(f_e(2x)-4f_e(x)) \tag{15}$$

for all $x \in X$. Setting $y = x$ in (5), we get

$$\begin{aligned} f_e((k+1)x)+f_e((k-1)x) &= k^2f_e(2x)+\frac{k^2(k^2-1)}{6}(f_e(2x)-4f_e(x)) \\ &\quad + 2(1-k^2)f_e(x) \end{aligned} \tag{16}$$

for all $x \in X$. Putting $y = kx$ in (5), we obtain

$$\begin{aligned} f_e(2kx) =& k^2(f_e((k+1)x)) + f_e((k-1)x)) + \frac{k^2(k^2-1)}{6}(f_e(2x) - 4f_e(x)) \\ &+ 2(1-k^2)f_e(kx) \end{aligned} \tag{17}$$

for all $x \in X$. Letting $y = 0$ in (13), we have

$$f_e(2kx) = k^2 f_e(2x) + \frac{k^2(k^2-1)}{12}(f_e(4x) - 4f_e(2x)) \tag{18}$$

for all $x \in X$. It follows from (17) and (18) that

$$\begin{aligned} \frac{k^2(k^2-1)}{12}(f_e(4x) - 4f_e(2x)) &= k^2(f_e((k+1)x)) + f_e((k-1)x)) \\ &+ \frac{k^2(k^2-1)}{6}(f_e(2x) - 4f_e(x)) + 2(1-k^2)f_e(kx) - k^2 f_e(2x) \end{aligned} \tag{19}$$

for all $x \in X$. Now, using (15), (16) and (19), we are lead to

$$\begin{aligned} &\frac{k^2(k^2-1)}{12}(f_e(4x) - 4f_e(2x)) \\ &= k^2[k^2 f_e(2x) + \frac{k^2(k^2-1)}{6}(f_e(2x) - 4f_e(x)) + 2(1-k^2)f_e(x)] \\ &\quad + 2(1-k^2)[k^2 f_e(x) + \frac{k^2(k^2-1)}{12}(f_e(2x) - 4f_e(x))] \\ &\quad + \frac{k^2(k^2-1)}{6}(f_e(2x) - 4f_e(x)) - k^2 f_e(2x) \end{aligned} \tag{20}$$

for all $x \in X$. Finally, if we compare (14) with (20), then we conclude that

$$f_e(2x+y) + f_e(2x-y) = 4f_e(x+y) + 4f_e(x-y) + 2(f_e(2x) - 4f_e(x)) - 6f_e(y) \tag{21}$$

for all $x, y \in X$. Replacing y by $2y$ in (21), we get

$$\begin{aligned} f_e(2x+2y) + f_e(2x-2y) &= 4f_e(x+2y) + 4f_e(x-2y) \\ &+ 2(f_e(2x) - 4f_e(x)) - 6f_e(2y) \end{aligned} \tag{22}$$

for all $x, y \in X$. Interchanging x with y in (21), we obtain

$$f_e(2y+x) + f_e(2y-x) = 4f_e(y+x) + 4f_e(y-x) + 2(f_e(2y) - 4f_e(y)) - 6f_e(x) \tag{23}$$

for all $x, y \in X$, which implies that

$$f_e(x+2y) + f_e(x-2y) = 4f_e(x+y) + 4f_e(x-y) + 2(f_e(2y) - 4f_e(y)) - 6f_e(x) \tag{24}$$

for all $x, y \in X$. It follows from (22) and (24) that

$$\begin{aligned} f_e(2(x+y)) - 16f_e(x+y) + f_e(2(x-y)) - 16f_e(x-y) \\ = 2(f_e(2x) - 16f_e(x)) + 2(f_e(2y) - 16f_e(y)) \end{aligned}$$

for all $x, y \in X$. This means that

$$g(x+y) + g(x-y) = 2g(x) + 2g(y)$$

for all $x, y \in X$. So the function $g : X \to Y$ defined by $g(x) = f_e(2x) - 16f_e(x)$ is quadratic.

To prove that $h : X \to Y$ defined by $h(x) = f_e(2x) - 4f_e(x)$ is quartic, we have to show that

$$h(2x+y) + h(2x-y) = 4h(x+y) + 4h(x-y) + 24h(x) - 6h(y) \tag{25}$$

for all $x, y \in X$. Replacing x and y by $2x$ and $2y$ in (21), respectively, we obtain

$$\begin{aligned} f_e(2(2x+y)) + f_e(2(2x-y)) &= 4f_e(2(x+y)) + 4f_e(2(x-y)) \\ &\quad + 2(f_e(4x) - 4f_e(2x)) - 6f_e(2y) \end{aligned} \tag{26}$$

for all $x, y \in X$. But, since $g(2x) = 4g(x)$ for all $x \in X$, with $g : X \to Y$ is the quadratic function defined above, thus we see that

$$f_e(4x) = 20f_e(2x) - 64f_e(x) \tag{27}$$

for all $x \in X$. Hence, according to (26) and (27), we get

$$\begin{aligned} f_e(2(2x+y)) + f_e(2(2x-y)) &= 4f_e(2(x+y)) + 4f_e(2(x-y)) \\ &\quad + 32(f_e(2x) - 4f_e(x)) - 6f_e(2y) \end{aligned} \tag{28}$$

for all $x, y \in X$. By multiplying both sides of (21) by 4, we get that

$$\begin{aligned} 4f_e(2x+y) + 4f_e(2x-y) &= 16f_e(x+y) + 16f_e(x-y) \\ &\quad + 8(f_e(2x) - 4f_e(x)) - 24f_e(y) \end{aligned} \tag{29}$$

for all $x, y \in X$. If we subtract the last equation from (28), we arrive at

$$\begin{aligned} &f_e(2(2x+y)) - 4f_e(2x+y) + f_e(2(2x-y)) - 4f_e(2x-y) \\ &\quad = 4(f_e(2(x+y)) - 4f_e(x+y)) + 4(f_e(2(x-y)) - 4f_e(x-y)) \\ &\quad\quad + 24(f_e(2x) - 4f_e(x)) - 6(f_e(2y) - 4f_e(y)) \end{aligned}$$

for all $x, y \in X$. This means that h satisfies the equation (25), so the function $h : X \to Y$ is quartic. But, since, $f_e(x) = \frac{1}{12}h(x) - \frac{1}{12}g(x)$ for all $x \in X$, there exist a unique symmetric bi-quadratic function $B_2 : X \times X \longrightarrow Y$ and a unique symmetric bi-additive function $B_1 : X \times X \longrightarrow Y$ such that $h(x) = 12B_2(x, x)$ and $g(x) = -12B_1(x, x)$ for all $x \in X$ (see [1, 14]). So

$$f_e(x) = B_2(x, x) + B_1(x, x) \tag{30}$$

for all $x \in X$. On the other hand, we show that the function $f_o : X \to Y$ is cubic.

It follows from (4) and $f_o(-x) = -f_o(x)$ that

$$\begin{aligned} f_o(x+ky) + f_o(x-ky) &= k^2 f_o(x+y) + k^2 f_o(x-y) + 2(1-k^2)f_o(x) \\ &\quad + \frac{k^2(k^2-1)}{6}(f_o(2y) - 8f_o(y)) \end{aligned} \tag{31}$$

for all $x, y \in X$. Putting $x = 0$ in (31), to get $f_o(2y) = 8f_o(y)$ for all $y \in X$, so we get from (31) that

$$f_o(x+ky) + f_o(x-ky) = k^2 f_o(x+y) + k^2 f_o(x-y) + 2(1-k^2)f_o(x) \tag{32}$$

for all $x, y \in X$. Setting $x = x - y$ in (32), we have

$$\begin{aligned} f_o(x+(k-1)y) + f_o(x-(k+1)y) &= k^2 f_o(x) + k^2 f_o(x-2y) \\ &\quad + 2(1-k^2)f_o(x-y) \end{aligned} \tag{33}$$

for all $x, y \in X$. Replacing y by $-y$ in (33), gives

$$\begin{aligned} f_o(x-(k-1)y)+f_o(x+(k+1)y) &= k^2 f_o(x)+k^2 f_o(x+2y) \\ &\quad + 2(1-k^2)f_o(x+y) \end{aligned} \tag{34}$$

for all $x, y \in X$. Adding (33) to (34), we obtain

$$\begin{aligned} f_o(x+(k-1)y)+f_o(x-(k-1)y)+f_o(x+(k+1)y)+f_o(x-(k+1)y) \\ = k^2(f_o(x+2y)+f_o(x-2y))+2k^2 f_o(x) \\ + 2(1-k^2)(f_o(x+y)+f_o(x-y)) \end{aligned} \tag{35}$$

for all $x, y \in X$. Setting $x = x + ky$ in (32), we get

$$\begin{aligned} f_o(x+2ky)+f_o(x) = k^2(f_o(x+(k-1)y)+f_o(x+(k+1)y))) \\ + 2(1-k^2)f_o(x+ky) \end{aligned} \tag{36}$$

for all $x, y \in X$. Replacing y by $-y$ in (36), we have

$$\begin{aligned} f_o(x-2ky)+f_o(x) = k^2(f_o(x-(k-1)y)+f_o(x-(k+1)y))) \\ + 2(1-k^2)f_o(x-ky) \end{aligned} \tag{37}$$

for all $x, y \in X$. Adding (36) to (37), one gets

$$\begin{aligned} & f_o(x+2ky)+f_o(x-2ky) \\ &= k^2[f_o(x+(k-1)y)+f_o(x+(k+1)y)+f_o(x-(k-1)y) \\ &\quad + f_o(x-(k+1)y)]+2(1-k^2)(f_o(x+ky)+f_o(x-ky))-2f_o(x) \end{aligned} \tag{38}$$

for all $x, y \in X$. Using (32), (35) and (38), we lead to

$$\begin{aligned} & f_o(x+2ky)+f_o(x-2ky) \\ &= 4k^2(1-k^2)(f_o(x+y)+f_o(x-y))+(6k^4-8k^2+2)f_o(x) \\ &\quad + k^4(f_o(x+2y)+f_o(x-2y)) \end{aligned} \tag{39}$$

for all $x, y \in X$. Replacing y by $2y$ in (32), we get

$$f_o(x+2ky)+f_o(x-2ky) = k^2(f_o(x+2y)+f_o(x-2y))+2(1-k^2)f_o(x) \tag{40}$$

for all $x, y \in X$. If we compare (39) with (40), then we conclude that

$$f_o(x+2y)+f_o(x-2y) = 4(f_o(x+y)+f_o(x-y))-6f_o(x) \tag{41}$$

for all $x, y \in X$. Replacing x by $2x$ in (41), gives

$$f_o(2(x+y))+f_o(2(x-y)) = 4(f_o(2x+y)+f_o(2x-y))-6f_o(2x) \tag{42}$$

for all $x, y \in X$, which by considering $f_o(2x) = 8f_o(x)$ and (42), gives

$$f_o(2x+y)+f_o(2x-y) = 2f_o(x+y)+2f_o(x-y)12f_o(x)$$

for all $x, y \in X$, this means that f_o is cubic. So

$$f_o(x) = C(x,x,x) \tag{43}$$

for all $x \in X$, that C is symmetric for each fixed one variable and is additive for fixed two variables [11].

Hence, according to (30) and (43), we obtain that

$$f(x) = f_e(x)+f_o(x) = B_2(x,x)+C(x,x,x)+B_1(x,x)$$

for all $x \in X$. The proof of the converse is trivial. □

3. Generalized Hyers–Ulam stability in non-Archimedean spaces

Throughout this section, assume that G is an additive group and X is a complete non-Archimedean space. Before taking up the main subject, for $f : G \times G \to X$, we define the difference operator

$$\begin{aligned} Df(x,y) =& f(x+ky) + f(x-ky) - k^2 f(x+y) - k^2 f(x-y) + 2(k^2-1)f(x) \\ &- \frac{k^2(k^2-1)}{6}(f(2y) + 2f(-y) - 6f(y)) \end{aligned}$$

$(k \in \mathbb{Z} - \{0, \pm 1\})$ for all $x, y \in G$.

Theorem 3.1. *Let $\varphi : G \times G \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|2|^{2n}} = 0 = \lim_{n\to\infty} \frac{1}{|2|^{2n}} \tilde{\varphi}(2^{n-1}x) \tag{44}$$

for all $x, y \in G$, and

$$\tilde{\varphi}_q(x) = \lim_{n\to\infty} \max\{\frac{1}{|2|^{2j}} \tilde{\varphi}(2^j x) : \ 0 \le j < n\} \tag{45}$$

exists for all $x \in G$, where

$$\begin{aligned} \tilde{\varphi}(x) := \frac{1}{|k^2(k^2-1)|} \max\{ &\max\{|12k^2|\varphi(x,x), |12(k^2-1)|\varphi(0,x)\} \\ &, \max\{|6|\varphi(0,2x), |12|\varphi(kx,x)\}\} \end{aligned} \tag{46}$$

for all $x \in G$. Suppose that an even function $f : G \to X$ with $f(0) = 0$ satisfies the inequality

$$\|Df(x,y)\| \le \varphi(x,y) \tag{47}$$

for all $x, y \in G$. Then there exist a quadratic function $Q : G \to X$ such that

$$\|f(2x) - 16f(x) - Q(x)\| \le \frac{1}{|2|^2} \tilde{\varphi}_q(x) \tag{48}$$

for all $x \in G$. If

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\{\frac{1}{|2|^{2j}} \tilde{\varphi}(2^j x) : \ i \le j < n+i\} = 0 \tag{49}$$

then Q is the unique quadratic function satisfying (48).

Proof. Interchanging x with y in (47) and then using the evenness of f, we obtain

$$\begin{aligned} \|f(kx+y) &+ f(kx-y) - k^2 f(x+y) - k^2 f(x-y) + 2(k^2-1)f(y) \\ &- \frac{k^2(k^2-1)}{6}(f(2x) - 4f(x))\| \le \varphi(y,x) \end{aligned} \tag{50}$$

for all $x \in G$. Setting $y = 0$ in (50), we have

$$\|2f(kx) - 2k^2 f(x) - \frac{k^2(k^2-1)}{6}(f(2x) - 4f(x))\| \le \varphi(0,x) \tag{51}$$

for all $x \in G$. Putting $y = x$ in (50), we get

$$\begin{aligned} \|f((k+1)x) + f((k-1)x) - k^2 f(2x) + 2(k^2-1)f(x) \\ - \frac{k^2(k^2-1)}{6}(f(2x) - 4f(x))\| \leq \varphi(x,x) \end{aligned} \tag{52}$$

for all $x \in G$. Replacing x and y by $2x$ and 0 in (50), respectively, we obtain

$$\|2f(2kx) - 2k^2 f(2x) - \frac{k^2(k^2-1)}{6}(f(4x) - 4f(2x))\| \leq \varphi(0,2x) \tag{53}$$

for all $x \in G$. Setting $y = kx$ in (50) and using the evenness of f, it follows that

$$\begin{aligned} \|f(2kx) - k^2 f((k+1)x) - k^2 f((k-1)x) + 2(k^2-1)f(kx) \\ - \frac{k^2(k^2-1)}{6}(f(2x) - 4f(x))\| \leq \varphi(kx,x) \end{aligned} \tag{54}$$

for all $x \in G$. It follows from (51)–(54) that

$$\begin{aligned} &\|f(4x) - 20f(2x) + 64f(x)\| \\ &\leq \frac{1}{|k^2(k^2-1)|} \max\{\max\{|12k^2|\varphi(x,x), |12(k^2-1)|\varphi(0,x)\} \\ &\qquad , \max\{|6|\varphi(0,2x), |12|\varphi(kx,x)\}\} \end{aligned} \tag{55}$$

for all $x \in G$. According to (46) and (55), we obtain

$$\|f(4x) - 20f(2x) + 64f(x)\| \leq \tilde{\varphi}(x) \tag{56}$$

for all $x \in G$. Let $g : X \to Y$ be a function defined by $g(x) := f(2x) - 16f(x)$ for all $x \in G$. From (56), we conclude that

$$\|g(2x) - 4g(x)\| \leq \tilde{\varphi}(x) \tag{57}$$

for all $x \in G$. This implies that

$$\|g(x) - \frac{g(2x)}{2^2}\| \leq \frac{1}{|2|^2}\tilde{\varphi}(x) \tag{58}$$

for all $x \in G$. Replacing x by $2^{n-1}x$ in (58), we have

$$\|\frac{g(2^{n-1}x)}{2^{2(n-1)}} - \frac{g(2^n x)}{2^{2n}}\| \leq \frac{1}{|2|^{2n}}\tilde{\varphi}(2^{n-1}x) \tag{59}$$

for all $x \in G$. It follows from (44) and (59) that the sequence $\{\frac{g(2^n x)}{2^{2n}}\}$ is Cauchy. Since X is complete, we conclude that $\{\frac{g(2^n x)}{2^{2n}}\}$ is convergent. So one can define the function $Q : X \to Y$ by $Q(x) := \lim_{n\to\infty} \frac{g(2^n x)}{2^{2n}}$ for all $x \in G$. It follows from (58) and (59) by using induction that

$$\|g(x) - \frac{g(2^n x)}{2^{2n}}\| \leq \frac{1}{|2|^2} \max\{\frac{1}{|2|^{2j}}\tilde{\varphi}(2^j x) : \ 0 \leq j < n\} \tag{60}$$

for all $n \in \mathbb{N}$ and all $x \in G$. Taking n to approach infinity in (60) and using (45) one gets (48). By (44) and (47), we obtain

$$\begin{aligned}\|DQ(x,y)\| &= \lim_{n\to\infty} \frac{1}{|2|^{2n}} \|Dg(2^n x, 2^n y)\| \\ &= \lim_{n\to\infty} \frac{1}{|2|^{2n}} \|Df(2^{n+1}x, 2^{n+1}y) - 16Df(2^n x, 2^n y)\| \\ &\le \lim_{n\to\infty} \frac{1}{|2|^{2n}} \max\{\|Df(2^{n+1}x, 2^{n+1}y)\|, |16|\ \|Df(2^n x, 2^n y)\|\} = 0\end{aligned}$$

for all $x, y \in G$. Therefore the function $Q : G \to X$ satisfies (4). Thus by Theorem 2.1, the function $x \rightsquigarrow Q(2x) - 16Q(x)$ is quadratic.

If Q' is another quadratic function satisfying (48), then

$$\begin{aligned}\|Q(x) - Q'(x)\| &= \lim_{i\to\infty} |2|^{-2i} \|Q(2^i x) - Q'(2^i x)\| \\ &\le \lim_{i\to\infty} |2|^{-2i} \max\{\ \|Q(2^i x) - f(2^i x)\|, \|f(2^i x) - Q'(2^i x)\|\ \} \\ &\le \frac{1}{|2|^2} \lim_{i\to\infty} \lim_{n\to\infty} \max\{\frac{1}{|2|^{2j}} \tilde{\varphi}(2^j x) :\ i \le j < n+i\} = 0\end{aligned}$$

for all $x \in G$. Therefore $Q = Q^{'}$. □

Theorem 3.2. *Let $\varphi : G \times G \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|2|^{4n}} = 0 = \lim_{n\to\infty} \frac{1}{|2|^{4n}} \tilde{\varphi}(2^{n-1}x) \tag{61}$$

for all $x, y \in G$, and

$$\tilde{\varphi}_v(x) = \lim_{n\to\infty} \max\{\frac{1}{|2|^{4j}} \tilde{\varphi}(2^j x) :\ 0 \le j < n\} \tag{62}$$

exists for all $x \in G$, with $\tilde{\varphi}(x)$ satisfies the equation (46) *for all $x \in G$. Suppose that an even function $f : G \to X$ with $f(0) = 0$ satisfies the inequality* (47) *for all $x, y \in G$. Then there exist a quartic function $V : G \to X$ such that*

$$\|f(2x) - 4f(x) - V(x)\| \le \frac{1}{|2|^4} \tilde{\varphi}_v(x) \tag{63}$$

for all $x \in G$. If

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\{\frac{1}{|2|^{4j}} \tilde{\varphi}(2^j x) :\ i \le j < n+i\} = 0 \tag{64}$$

then V is the unique quartic function satisfying (63).

Proof. Similar to the proof Theorem 3.1, we have

$$\|f(4x) - 20f(2x) + 64f(x)\| \le \tilde{\varphi}(x) \tag{65}$$

for all $x \in G$. Let $h : X \to Y$ be a function defined by $h(x) := f(2x) - 4f(x)$ for all $x \in G$. From (65), we conclude that

$$\|h(2x) - 16h(x)\| \le \tilde{\varphi}(x) \tag{66}$$

for all $x \in G$. Which implies that

$$\|h(x) - \frac{h(2x)}{2^4}\| \le \frac{1}{|2|^4} \tilde{\varphi}(x) \tag{67}$$

for all $x \in G$. Replacing x by $2^{n-1}x$ in (67), we have

$$\|\frac{h(2^{n-1}x)}{2^{4(n-1)}} - \frac{h(2^n x)}{2^{4n}}\| \leq \frac{1}{|2|^{4n}}\tilde{\varphi}(2^{n-1}x) \tag{68}$$

for all $x \in G$. It follows from (61) and (68) that the sequence $\{\frac{h(2^n x)}{2^{4n}}\}$ is Cauchy. Since X is complete, we conclude that $\{\frac{h(2^n x)}{2^{4n}}\}$ is convergent. So one can define the function $V : X \to Y$ by $V(x) := \lim_{n\to\infty} \frac{h(2^n x)}{2^{4n}}$ for all $x \in G$. It follows from (67) and (68) by using induction that

$$\|h(x) - \frac{h(2^n x)}{2^{4n}}\| \leq \frac{1}{|2|^4} \max\{\frac{1}{|2|^{4j}}\tilde{\varphi}(2^j x) : \ 0 \leq j < n\} \tag{69}$$

for all $n \in \mathbb{N}$ and all $x \in G$. Taking n to approach infinity in (69) and using (62) one gets (63). By (61) and (47), we obtain

$$\begin{aligned}\|DV(x,y)\| &= \lim_{n\to\infty} \frac{1}{|2|^{4n}}\|Dh(2^n x, 2^n y)\| \\ &= \lim_{n\to\infty} \frac{1}{|2|^{4n}}\|Df(2^{n+1}x, 2^{n+1}y) - 4Df(2^n x, 2^n y)\| \\ &\leq \lim_{n\to\infty} \frac{1}{|2|^{4n}} \max\{\|Df(2^{n+1}x, 2^{n+1}y)\|, |4| \ \|Df(2^n x, 2^n y)\|\} = 0\end{aligned}$$

for all $x, y \in G$. Therefore the function $V : G \to X$ satisfies (4). Thus by Theorem 2.1, the function $x \rightsquigarrow V(2x) - 4V(x)$ is quartic. The rest of the proof is similar to the proof of Theorem 3.1. □

Theorem 3.3. *Let $\varphi : G \times G \to [0, \infty)$ be a function satisfies* (44) *for all $x, y \in G$, and the limit*

$$\lim_{n\to\infty} \max\{\frac{1}{|2|^{2j}}\tilde{\varphi}(2^j x) : \ 0 \leq j < n\}$$

exists for all $x \in G$, with $\tilde{\varphi}(x)$ satisfies the equation (46) *for all $x \in G$. Suppose that an even function $f : G \to X$ with $f(0) = 0$ satisfies the inequality* (47) *for all $x, y \in G$. Then there exist a quadratic function $Q : G \to X$ and a quartic function $V : G \to X$ such that*

$$\|f(x) - Q(x) - V(x)\| \leq \frac{1}{|48|} \max\{\tilde{\varphi}_q(x), \frac{1}{|2|^2}\tilde{\varphi}_v(x)\} \tag{70}$$

for all $x \in G$, where $\tilde{\varphi}_q(x)$ and $\tilde{\varphi}_v(x)$ are defined as in Theorems 3.1 *and* 3.2*. Moreover, if* (49) *holds, then Q is the unique quadratic function and V is the unique quartic function satisfying* (70).

Proof. By Theorem 3.1 and 3.2, there exists a quadratic function $Q_0 : G \to X$ and a quartic function $V_0 : G \to X$ such that

$$\|f(2x) - 16f(x) - Q_0(x)\| \leq \frac{1}{|2|^2}\tilde{\varphi}_q(x)$$

$$\|f(2x) - 4f(x) - V_0(x)\| \leq \frac{1}{|2|^4}\tilde{\varphi}_v(x)$$

for all $x \in G$. So we obtain (70) by letting $Q(x) = -1/12Q_0(x)$ and $V(x) = 1/12V_0(x)$ for all $x \in G$.

To prove the uniqueness property of Q and V, let $Q', V' : X \to Y$ be another quadratic and quartic functions satisfying (70). Let $\overline{Q} = Q - Q'$ and $\overline{V} = V - V'$. Hence

$$\begin{aligned}\|\overline{Q}(x) + \overline{V}(x)\| &\leq \max\{\|f(x) - Q(x) - V(x)\| + \|f(x) - Q'(x) - V'(x)\|\} \\ &\leq \frac{1}{|48|} \max\{\tilde{\varphi}_q(x), \frac{1}{|2|^2}\tilde{\varphi}_v(x)\}\end{aligned}$$

for all $x \in G$. Since

$$\begin{aligned}&\lim_{i\to\infty}\lim_{n\to\infty} \max\{\frac{1}{|2|^{2j}}\tilde{\varphi}(2^j x) : \ i \leq j < n+i\} \\ &= 0 = \lim_{i\to\infty}\lim_{n\to\infty} \max\{\frac{1}{|2|^{4j}}\tilde{\varphi}(2^j x) : \ i \leq j < n+i\}\end{aligned}$$

for all $x \in G$. So

$$\lim_{n\to\infty} \frac{1}{|2|^{4n}} \|\overline{Q}(2^n x) + \overline{V}(2^n x)\| = 0$$

for all $x \in X$. Therefore, we get $\overline{V} = 0$ and then $\overline{Q} = 0$, and the proof is complete. □

Theorem 3.4. *Let $\varphi : G \times G \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|2|^{3n}} = 0 \tag{71}$$

for all $x, y \in G$ and

$$\tilde{\varphi}_c(x) = \lim_{n\to\infty} \max\{\frac{1}{|2|^{3j}}\varphi(0, 2^j x) : \ 0 \leq j < n\} \tag{72}$$

for all $x \in G$, exists. Suppose that $f : G \to X$ is an odd function satisfying (47) for all $x, y \in G$. Then there exist a cubic function $C : G \to X$ such that

$$\|f(x) - C(x)\| \leq |\frac{3}{4k^2(k^2-1)}|\tilde{\varphi}_c(x) \tag{73}$$

for all $x \in G$. If

$$\lim_{i\to\infty}\lim_{n\to\infty} \max\{\frac{1}{|2|^{3j}}\varphi(0, 2^j x) : \ i \leq j < n+i\} = 0$$

then C is the unique cubic function satisfying (73).

Proof. Putting $x = 0$ in (47) and then using oddness of f, we get

$$\|\frac{k^2(k^2-1)}{6}(f(2y) - 8f(y))\| \leq \varphi(0, y)$$

for all $y \in G$. Which implies that

$$\|f(x) - \frac{f(2x)}{2^3}\| \leq |\frac{3}{4k^2(k^2-1)}|\varphi(0, x) \tag{74}$$

for all $x \in G$. Replacing x by $2^{n-1}x$ in (74), we have

$$\|\frac{f(2^{n-1}x)}{2^{3(n-1)}} - \frac{f(2^n x)}{2^{3n}}\| \leq |\frac{3}{2^{3n-1}k^2(k^2-1)}|\varphi(0, 2^{n-1}x) \tag{75}$$

for all $x \in G$. It follows from (71) and (75) that the sequence $\{\frac{f(2^n x)}{2^{3n}}\}$ is Cauchy. Since X is complete, we conclude that $\{\frac{f(2^n x)}{2^{3n}}\}$ is convergent. So one can define the function $C : X \to Y$ by $C(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^{3n}}$ for all $x \in G$. It follows from (74) and (75) by using induction that

$$\|f(x) - \frac{f(2^n x)}{2^{3n}}\| \leq |\frac{3}{4k^2(k^2-1)}| \max\{\frac{1}{|2|^{3i}}\varphi(0, 2^i x) : \ 0 \leq i < n\} \tag{76}$$

for all $n \in \mathbb{N}$ and all $x \in G$. By taking n to approach infinity in (76) and using (72) one obtains (73). By (71) and (47), one gets

$$\|DC(x,y)\| = \lim_{n\to\infty} \frac{1}{|2|^{3n}}\|Df(2^n x, 2^n y)\| \leq \lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|2|^{3n}} = 0$$

for all $x, y \in G$. Therefore the function $C : G \to X$ satisfies (4). Thus by Theorem 2.1, the function $x \rightsquigarrow C(x)$ is cubic. The rest of the proof is similar to the proof of Theorem 3.1. □

Now, we are ready to prove the main theorem concerning the stability problem for the equation (4).

Theorem 3.5. *Let $\varphi : G \times G \to [0, \infty)$ be a function satisfies (44) for all $x, y \in G$, and the limits*

$$\lim_{n\to\infty} \max\{\frac{1}{|2|^{2j}}\tilde{\varphi}(2^j x) : \ 0 \leq j < n\}, \qquad \lim_{n\to\infty} \max\{\frac{1}{|2|^{3j}}\varphi(0, 2^j x) : \ 0 \leq j < n\}$$

exists for all $x \in G$, with $\tilde{\varphi}(x)$ satisfies the equation (46) for all $x \in G$. Suppose that a function $f : G \to X$ with $f(0) = 0$ satisfies the inequality (47) for all $x, y \in G$. Then there exist a quadratic function $Q : G \to X$, a cubic function $C : G \to X$ and a quartic function $V : G \to X$ such that

$$\begin{aligned} &\|f(x) - Q(x) - C(x) - V(x)\| \\ &\leq \frac{1}{|2|^3} \max\{|\frac{3}{k^2(k^2-1)}| \max\{\tilde{\varphi}_c(x), \tilde{\varphi}_c(-x)\} \\ &\qquad , \frac{1}{|12|} \max\{\max\{\tilde{\varphi}_q(x), \frac{1}{|2|^2}\tilde{\varphi}_v(x)\}, \max\{\tilde{\varphi}_q(-x), \frac{1}{|2|^2}\tilde{\varphi}_v(-x)\}\}\} \end{aligned} \tag{77}$$

for all $x \in G$, where $\tilde{\varphi}_q(x)$, $\tilde{\varphi}_c(x)$ and $\tilde{\varphi}_v(x)$ are defined as in Theorems 3.1, 3.2 and 3.4. If

$$\begin{aligned} &\lim_{i\to\infty} \lim_{n\to\infty} \max\{\frac{1}{|2|^{2j}}\tilde{\varphi}(2^j x) : \ 0 \leq j < n\} \\ &\qquad = 0 = \lim_{i\to\infty} \lim_{n\to\infty} \max\{\frac{1}{|2|^{3j}}\varphi(0, 2^j x) : \ i \leq j < n+i\} \end{aligned}$$

then Q is the unique quadratic function, C is the unique cubic function and V is the unique quartic function satisfying (77).

Proof. Let $f_o(x) = \frac{1}{2}(f(x) - f(-x))$ for all $x \in G$. Then

$$\|Df_o(x,y)\| \leq \frac{1}{|2|} \max\{\varphi(x,y), \varphi(-x,-y)\}$$

for all $x, y \in G$. From Theorem 3.4, it follows that there exists a unique cubic function $C : G \to X$ satisfying

$$\|f_o(x) - C(x)\| \leq |\frac{3}{8k^2(k^2-1)}| \max\{\tilde{\varphi}_c(x), \tilde{\varphi}_c(-x)\} \tag{78}$$

for all $x \in G$. Also, let $f_e(x) = \frac{1}{2}(f(x) + f(-x))$ for all $x \in G$. Then

$$\|Df_e(x,y)\| \leq \frac{1}{|2|} \max\{\varphi(x,y), \varphi(-x,-y)\}$$

for all $x, y \in G$. From Theorem 3.3, it follows that there exist a quadratic function $Q : G \to X$ and a quartic function $V : G \to X$ satisfying

$$\begin{aligned} &\|f_e(x) - Q(x) - V(x)\| \\ &\leq \frac{1}{|96|} \max\{\max\{\tilde{\varphi}_q(x), \frac{1}{|2|^2}\tilde{\varphi}_v(x)\}, \max\{\tilde{\varphi}_q(-x), \frac{1}{|2|^2}\tilde{\varphi}_v(-x)\}\} \end{aligned} \tag{79}$$

for all $x \in G$. Hence, (77) follows from (78) and (79). The rest of the proof is trivial. □

REFERENCES

[1] *J. Aczl and J. Dhombres*,Functional Equations in Several Variables, Cambridge University Press, 1989.

[2] *T. Aoki*, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan **2** (1950), 64–66.

[3] *G. Bercu and M. Postolache*, Class of self–concordant functions on Riemannian manifolds, Balkan J. Geom. Appl., **14** (2009), No. 2, 13–20.

[4] *J.K. Chung and P.K. Sahoo*, On the general solution of a quartic functional equation, Bull. Korean Math. Soc. **40** (2003), no. 4, 565–576.

[5] *P. Czerwik*, Functional Equations and Inequalities in Several Variables, World Scientific Publishing Company, New Jersey, Hong Kong, Singapore, London, 2002.

[6] *M. Eshaghi Gordji and H. Khodaei*, Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi–Banach spaces, Nonlinear Analysis.-TMA **71** (2009), 5629–5643.

[7] *M. Eshaghi Gordji and M. Bavand Savadkouhi*, Stability of cubic and quartic functional equations in non-Archimedean spaces, Acta Applicandae Mathematica, **106**, No 3 (2009), DOI: 10.1007/s10440–009–9512–7.

[8] *P. Găvruta*,A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. **184** (1994), 431-436.

[9] *K. Hensel*, Uber eine neue Begrundung der Theorie der algebraischen Zahlen, Jahresber. Deutsch. Math. Verein, **6** (1897), 83-88.

[10] *D.H. Hyers*, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. **27** (1941), 222–224.

[11] *K.W. Jun and H.M. Kim*, The generalized Hyers–Ulam–Rassias stability of a cubic functional equation, J. Math. Anal. Appl. **274** (2002), 867–878.

[12] *Pl. Kannappan*,Quadratic functional equation and inner product spaces, Results Math. **27** (1995), 368–372.

[13] *A. Khrennikov*, Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models, Kluwer Academic Publishers, Dordrecht, 1997.

[14] *S.H. Lee, S.M. Im and I.S. Hawng*,Quartic functional equation , J. Math. Anal. Appl. **307** (2005), 387–394.

[15] *M.S. Moslehian and T.M. Rassias*, Stability of functional equations in non-Archimedean space, Applicable Analysis and Discrete Mathematics, **1** (2007), 325-334.

[16] *Th.M. Rassias*, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[17] *Th.M. Rassias*, On the stability of functional equations in Banach spaces, J. Math. Anal. Appl. **251** (2000), 264–284.

[18] *Th.M. Rassias*, On the stability of functional equations and a problem of Ulam, Acta Appl. Math. **62** (2000), 23–130.

[19] *Th.M. Rassias and P. Šemrl*, On the Hyers-Ulam stability of linear mappings, J. Math. Anal. Appl. **173** (1993), 325–338.

[20] *A. M. Robert*, A Course in p-adic Analysis, Springer-Verlag, New York, 2000.

[21] *S.M. Ulam*, Problems in Modern Mathematics, Chapter VI, science Editions., Wiley, New York, 1964.

[22] *A.C.M. Van rooij*, Non-Archimedean functional analysis, in: Monographs and Textbooks in Pure and Applied Math, **51**, Marcel Dekker, New York, 1978.

[23] *V.S. Vladimirov, I.V. Volovich and E.I. Zelenov*, p–adic Analysis and Mathematical Physics, World Scientific, 1994.