COMPARATIVE ANALYSIS OF ORTHOTROPIC PLATES

Iuliana SPRINȚU¹, Ion FUIOREA²

In this article are proposed some new mathematical solutions to the problem in displacements and corresponding modal analysis for thin orthotropic rectangular plates, having clamped edges. The considered reasons for the solutions were to exactly satisfy the boundary conditions and are relative to those found in the literature. The theoretical results obtained are compared with the experimental ones, aiming to complete an analysis.

Key words: thin plates, orthotropic, analytical model, clamped edges

1. Introduction

Composites are widely used in industry becoming an essential part of today’s materials as they offer advantages such as low weight, corrosion resistance, high fatigue strength, faster assembly, etc. Composite materials are heterogeneous and being the combination of different constituent materials can offer the desired combination of low weight, stiffness and strength. Nowadays, the knowledge has advanced to a level that materials can be tailored to exhibit certain, required properties. At the same time, the fact that these materials are composed of various, sometimes very different constituents, make their mechanical behaviour complex. This observation holds with respect to the deformation, failure or damage behavior. It is a challenge to develop predictive methods that can capture this complex mechanical behaviour, either using analytical tools [1], [2], [3], [4], or using numerical methods, the finite element method being the most

¹ Asist. univ. drd. Academia Tehnică Militară, București, România, sprintui@yahoo.com
² Prof. dr. ing. Ion Fuiorea, S.C. STRAERO S.A., București, România
widespread among the latter. There are also numerical approaches of applications involving composites, especially regarding impact problems by SPH [5], [6].

However, analytical models and solutions remain a prospective tool, even they are obtained generally for quite simple or particular cases of geometries and/or loadings. They are still very useful to validate a numerical solution, they are less expensive in time and calculation volume than the numerical ones, they can emphasize how the solution depends on data, etc.

This paper presents analytical solutions for the bending deformation of rectangular orthotropic composite plates. The models are based on the classical laminated plate theory [3], [4].

The considered reasons for the solutions were to exactly satisfy the boundary conditions and to verify as close as possible the differential equation of the plate. The weighted residue method was considered to optimise the chosen analytical solutions. Interesting evaluations were performed for different types of functions, especially with respect to the orthotropic answer of the plate. The purposed solutions were compared with those obtained by Reddy in [3]. Finally, the solutions were critically analysed considering a FEM solution and experimental data.

Thorough comparison between analytical solutions, numerical results and experimental data reveals a good agreement of the results.

2. Assumptions

The following assumptions are used in the analysis of thin plate model:

i) The constitutive law is orthotropic elasticity.

ii) Strain-displacement relation is linear, i.e. geometrical linearity.

The classical laminated plate theory is applied considering Kirchhoff-Love hypothesis:

iii) The inextensibility of normal is imposed, implying that during deformation the normal to the median of the plate remain straight, i.e. the transverse displacement is independent of the transverse (thickness) coordinate.

iv) During deformation the thickness plate remains constant, equivalent with $\varepsilon_z = 0$.

v) During deformation the transverse normals remain perpendicular to the midsurface, i.e. $\varepsilon_{xz} = 0, \varepsilon_{yz} = 0$. 
3. Orthotropic plate equations

For an orthotropic elastic thin rectangular plate subjected to an uniform distributed pressure \( p \) on the bottom face, plate with thickness \( h \), length \( a \) and width \( b \), the orthotropic elastic constitutive equations is:

\[
\sigma = \bar{Q} \cdot \varepsilon,
\]

where \( \bar{Q} = R(\theta) \cdot Q, \quad R(\theta) \in M_{6 \times 4}(\mathbb{R}) \), is the matrix that defines the rotation angle fibers \( \theta \) to the axis \( O_x \). \( Q = (Q_{11}, Q_{22}, Q_{12}, Q_{66})^T \) and

\[
Q_{11} = \frac{E_1}{1 - \nu_{12} \cdot \nu_{21}}, \quad Q_{22} = \frac{E_2}{1 - \nu_{12} \cdot \nu_{21}}, \quad Q_{12} = \frac{\nu_{21} \cdot E_1}{1 - \nu_{12} \cdot \nu_{21}}, \quad Q_{66} = G_{12},
\]

where \( E_1, E_2 \) are elasticity moduli in the longitudinal and transversal directions, respectively, \( G_{12} \) is the shear modulus in the plane of the ply, and \( \nu_{12} \) is the Poisson coefficient.

Replacing constitutive equation in the equilibrium equation and according to the hypothesis (i)-(v) it is obtained the partial differential equation for \( w \) (the displacement on \( O_z \) direction):

\[
\bar{Q}_{11} \frac{\partial^4 w}{\partial x^4} + \bar{Q}_{22} \frac{\partial^4 w}{\partial y^4} + 2 \left( \bar{Q}_{12} + 2 \cdot \bar{Q}_{66} \right) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4 \cdot \bar{Q}_{26} \frac{\partial^2 w}{\partial x \partial y^3} + 4 \cdot \bar{Q}_{16} \cdot \frac{\partial^4 w}{\partial x^3 \partial y} = 12 \cdot \frac{p}{h^3}.
\]

(1)

In particular case, when \( O_x \)-axis is oriented along the fiber direction, \( \theta = 0^\circ \), equation (1) becomes:

\[
Q_{11} \frac{\partial^4 w}{\partial x^4} + Q_{22} \frac{\partial^4 w}{\partial y^4} + 2 \cdot (Q_{12} + 2 \cdot G_{12}) \frac{\partial^4 w}{\partial x^2 \partial y^2} = 12 \cdot \frac{p}{h^3}.
\]

(2)

For modal analysis, in orthotropic plate equations as a uniformly distributed load \( \bar{p} \) is considered inertial force, resulting equation corresponding free vibrations:

\[
\frac{h^3}{12} \cdot A \ddot{w} + m \frac{\partial^2 w}{\partial t^2} = 0,
\]

(3)

where \( A = Q_{11} \frac{\partial^4}{\partial x^4} + Q_{22} \frac{\partial^4}{\partial y^4} + 2 \cdot (Q_{12} + 2 \cdot Q_{66}) \frac{\partial^4}{\partial x^2 \partial y^2} \) for \( \theta = 0^\circ \) and
Comparative analysis of orthotropic plates

\[ A = \tilde{Q}_{11} \frac{\partial^4}{\partial x^4} + \tilde{Q}_{22} \frac{\partial^4}{\partial y^4} + 2(\tilde{Q}_{12} + 2\tilde{Q}_{66}) \frac{\partial^4}{\partial x^2 \partial y^2} + 4\tilde{Q}_{16} \frac{\partial^4}{\partial x^3 \partial y} + 4\tilde{Q}_{26} \frac{\partial^4}{\partial x \partial y^3} \quad \text{for} \quad \theta \neq 0^\circ. \]

4. Mathematical solutions

For a rectangular orthotropic plate with clamped edges, must be solved equations (2) for \( \theta = 0^\circ \), respectively (1), for \( \theta \neq 0^\circ \), with boundary conditions:

\[
\begin{align*}
\left\{ \begin{array}{l}
w(0, y) = w(a, y) = 0, \quad \frac{\partial w}{\partial x}(0, y) = \frac{\partial w}{\partial x}(a, y) \\
w(x, 0) = w(x, b) = 0, \quad \frac{\partial w}{\partial y}(x, 0) = \frac{\partial w}{\partial y}(x, b) = 0.
\end{array} \right.
\tag{4}
\end{align*}
\]

Let us mention that Reddy proposes solution for equation (2) with boundary conditions (4) as [3],[7]:

\[
w(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \left( \frac{x}{a} \right)^{i+1} \left( \frac{y}{b} \right)^{j+1} \left( 1 - \frac{x}{a} \right)^2 \left( 1 - \frac{y}{b} \right)^2.
\tag{5}
\]

For this solution is observed that the terms corresponding to the plate edges \( x = a \) and \( y = b \) is not included in the sum.

In this article, using Ritz method and compared to Reddy's solution, it is proposed solution [8],[9]:

\[
w(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \left( \frac{x}{a} \right)^{i+1} \left( \frac{y}{b} \right)^{j+1} \left( 1 - \frac{x}{a} \right)^{i+1} \left( 1 - \frac{y}{b} \right)^{j+1}.
\tag{6}
\]

The coefficients \( c_{ij} \) will be determined using the weighted residue method.

According to this method, the following linear system is solved:

\[
\iint_D \left[ A(w(x, y)) - \frac{12p}{h^3} \right] \Phi_{ij}(x, y) \, dx \, dy = 0, \quad i = \overline{1,n}, \quad j = \overline{1,m}
\]

where \( \Phi_{ij} = \left( \frac{x}{a} \right)^{i+1} \left( \frac{y}{b} \right)^{j+1} \left( 1 - \frac{x}{a} \right)^{i+1} \left( 1 - \frac{y}{b} \right)^{j+1} \), \( i = \overline{1,n}, \quad j = \overline{1,m} \),

are weighting functions (Galerkin method) and \( D \) is the domain of the composite plate.

For a comparative analysis between solutions (5) and (6), it is considered a thin rectangular plate with \( a = 300 \text{mm}, \quad b = 200 \text{mm}, \quad h = 1.45 \text{mm} \),
Comparative analysis of orthotropic plates

\[ E_1 = 22051 MPa, \quad E_2 = 18512 MPa, \quad G_{12} = 8642 MPa, \quad v_{12} = 0.071, \]
subjected to a uniform pressure \( p = 0.00419 MPa \), using Maple we get the maximum value of the deflection \( w \left( \frac{a}{2}, \frac{b}{2} \right) = 3.053 mm \), for both solutions.

Note that in particular, for an isotropic thin plate, of 300×300×1.45 with \( E = 2205 MPa \), \( v = 0.071 \), subjected to a uniform pressure \( p = 0.00419 MPa \), the maximum value of the deflection using the new solution (6), is

\[ w \left( \frac{a}{2}, \frac{b}{2} \right) = 0.0012653 \cdot \frac{p \cdot a^4}{D}, \quad \text{where} \quad D = \frac{E \cdot h^3}{12 \left(1 - \nu^2\right)}. \]

This result is high close of the exact solution obtained by Timoshenko and Woinowsky,

\[ w \left( \frac{a}{2}, \frac{b}{2} \right) = 0.00126 \cdot \frac{p \cdot a^4}{D}. \]

In the same time, using Reddy’s solution, it result

\[ w \left( \frac{a}{2}, \frac{b}{2} \right) = 0.00133 \cdot \frac{p \cdot a^4}{D}. \]

Note that a good agreement between the two solutions (5) and (6), but a faster convergence is obtained using solution (6).

In case for angle \( \theta \neq 0^\circ \), must be solved equation (1) with boundary conditions (4).

In this article it is proposed a solution of form:

\[ w(x, y) = w_0(x, y) + \sum_{i=1}^{r} \sum_{j=1}^{s} k_{ij} \cdot f_i(x, \theta) \cdot g_j(y, \theta) \quad (7) \]

where \( w_0 \) is the proposed solution (6), and the last terms have been added to provide asymmetric solution and to check conditions on board outline.

For \( r = s = 1 \),

\[ f_1(x, \theta) = \tan \theta \cdot \left( \frac{x}{a} \right)^2 \cdot \left(1 - \frac{x}{a}\right)^3, \quad g_1(y, \theta) = \left( \frac{y}{b} \right)^2 \cdot \left(1 - \frac{y}{b}\right)^3 \]

\[ f_2(x, \theta) = \tan \theta \cdot \left( \frac{x}{a} \right)^2 \cdot \left(1 - \frac{x}{a}\right)^3, \quad g_2(y, \theta) = \left( \frac{y}{b} \right)^3 \cdot \left(1 - \frac{y}{b}\right)^2. \]

For the same orthotropic plate, with \( \theta = 20^\circ \), subjected to a uniform pressure \( p = 0.00419 MPa \), using Maple we get \( w \left( \frac{a}{2}, \frac{b}{2} \right) = 3.0138 mm \).
The proposed analytical solution (7), for orthotropic rectangular plate with clamped edges, when \( \theta \neq 0^\circ \), has the same theory as presented in [9], except that it starts from the solution (6), not from (5) proposed by Reddy in [3].

Note that both new solutions, (6) and (7), were critically analysed considering a FEM solution (SHELL63) and experimental data, using an experimental device.

In fig. 1 and table 1 it is presented the comparative results between the experimental measurements, numerical and analytical results, for the orthotropic plate of dimensions mentioned, having clamped edges, \( \theta = 0^\circ \), subjected to the uniform pressure \( p = 0.00419 \text{ MPa} \).

In fig. 2 and table 2 shows the comparative results for the orthotropic plate having clamped edges, for \( \theta = 20^\circ \), subjected to the uniform pressure \( p = 0.00419 \text{ MPa} \).

Fig. 1 (orthotropic plate with \( \theta = 0^\circ \), for \( y = 95 \text{ mm} \))

Fig. 2 (orthotropic plate with \( \theta = 20^\circ \), for \( y = 95 \text{ mm} \))
Comparative analysis of orthotropic plates

To perform modal analysis, to solve equation (3) where the differential operator is

\[ A = Q_{11} \frac{\partial^4}{\partial x^4} + Q_{22} \frac{\partial^4}{\partial y^4} + 2(Q_{12} + 2 \cdot Q_{66}) \frac{\partial^4}{\partial x^2 \partial y^2}, \]

with boundary conditions (4), in this article it is considered a solution of form

\[ w(x, y) = \sum_{n \geq 1} \sum_{m \geq 1} c_{nm} \cdot w_0 (x, y) \cdot \sin\left(\frac{n\pi x}{a}\right) \cdot \sin\left(\frac{m\pi y}{b}\right), \]

where

\[ w_0 (x, y) = \left(\frac{x}{a}\right)^2 \left(1 - \frac{x}{a}\right)^2 \left(\frac{y}{b}\right)^2 \left(1 - \frac{y}{b}\right)^2. \]

For \( m = n = 1 \) it is result frequency 1, \( \nu_1 = 149.99 \text{ Hz} \).

In the same time, for \( n = 2, m = 1, \) results \( \nu_2 = 246.465\text{Hz}, \) for \( n = 1, m = 2, \) \( \nu_3 = 367.395\text{Hz}, \) for \( n = 3, m = 1, \) \( \nu_4 = 382.841\text{Hz} \) and for \( n = 2, m = 2, \) results \( \nu_5 = 453.596\text{Hz}. \)

These results were critically analysed considering FEM solution (SHELL63) and experimental data.

In Table 3 it is presented the comparative results between the experimental measurements, numerical and analytical results, for the orthotropic plate of dimension mentioned, having clamped edges, with \( \theta = 0^\circ \).

<table>
<thead>
<tr>
<th>Frequency</th>
<th>( \nu_1 )</th>
<th>( \nu_2 )</th>
<th>( \nu_3 )</th>
<th>( \nu_4 )</th>
<th>( \nu_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytic (a)</td>
<td>149.99Hz</td>
<td>246.465Hz</td>
<td>367.395Hz</td>
<td>382.841Hz</td>
<td>453.596Hz</td>
</tr>
<tr>
<td>FEM (b)</td>
<td>153.59Hz</td>
<td>243.21Hz</td>
<td>369.50Hz</td>
<td>396.07Hz</td>
<td>447.71Hz</td>
</tr>
<tr>
<td>Experim. (c)</td>
<td>149Hz</td>
<td>226Hz</td>
<td>334Hz</td>
<td>379Hz</td>
<td>430Hz</td>
</tr>
<tr>
<td>Diff. (%a-b)</td>
<td>-2.4 %</td>
<td>1.32 %</td>
<td>-0.572 %</td>
<td>-3.455 %</td>
<td>1.297 %</td>
</tr>
<tr>
<td>Diff. (%a-c)</td>
<td>0.66 %</td>
<td>8.3 %</td>
<td>9.08 %</td>
<td>1.003 %</td>
<td>5.201 %</td>
</tr>
</tbody>
</table>
Comparative analysis of orthotropic plates

Small differences in results validate the proposed mathematical solutions in this article.

To perform experimental measurements was used the device in [7],[9]. In this article, a comprehensive analysis on the influence of reinforcement fibers, \( \theta \), only one big orthotropic plate was manufactured at STRAERO SA, having a thickness of 1.45 mm, which was cut in two rectangular plates of size (200/300), one with an angle \( \theta = 0^\circ \), another with \( \theta = 20^\circ \).

5. Conclusions

The analitical solutions can offer accurate and rapid solutions with respect to the numerical ones enough close to the experimental tests.

The additional terms of the solution can cover both different types of boundary conditions as well as different values of the anisotropy directions.

The testing device can offer accurate loading system and continuous measurements along different curves chosen upon the free surface of the plate.

Solving rectangular plates is important, their corresponding theory can be extended by a conform transformation to any surface.

REFERENCES